# Asymptotic wave-wave processes beyond cascading in quadratic nonlinear optical materials

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(Received 30 September 1994; revised manuscript received 29 March 1995)

The method of multiple scales is used to derive several different systems of evolution equations for multiple interacting waves propagating in a strongly dispersive, weakly quadratically nonlinear optical material. Several two- and three-wave signaling problems are discussed. Among the problems discussed are the interaction between a low-frequency field and the optical frequency field and between the optical frequency field and its second-harmonic field. In the efficient phase-matching regime, three-wave-mixing equations are obtained where quadratic nonlinearities dominate. Here, methods are discussed for cascading second-order nonlinearities to obtain intensity-dependent effects. For the large-phase-mismatch regime, cross-phase-modulation equations, analogous to fiber optics, are obtained where cubic nonlinearities dominate, and intensity-dependent modulations beyond cascading are obtained. Finally, the three-interacting- (sum frequency) wave problem is examined for small and large asymptotic phase-mismatch regimes. Analytical solutions to the derived evolution equations are given.

PACS number(s): 42.70.Nq, 42.81.Dp

## I. INTRODUCTION

Various asymptotic far-field evolution equations due to wave-wave interactions may be obtained for multiple input waves propagating in a quadratically nonlinear optical material. Traditionally, for three input waves, three-wave resonance equations are obtained for quadratic nonlinear materials that describe various well known $\chi^{(2)}$ wave processes such as second-harmonic generation and sum frequency mixing [1–3]. In optics, the three-wave resonance equations are usually derived from the slowly varying envelope approximation (SVEA), a popular perturbation method based on a Fourier series type expansion [4]. However, SVEA cannot be used to derive certain types of asymptotic far-field equations that arise from higher-order perturbation theory because it has no explicitly defined perturbation parameter. A more robust method that is valid to any perturbation order is the method of multiple scales (MMS). This more robust method is utilized in this paper to derive nontraditional asymptotic wave-wave interaction processes for quadratically nonlinear ($\chi^{(2)}$) optical materials. For example, using MMS we derive various intensity-dependent modulation processes for $\chi^{(2)}$ materials that are traditionally associated with $\chi^{(3)}$ materials.

MMS is a self-consistent perturbation method developed by Cole [5], Sturrock [6], and Sandri [7] that expands both the field and independent variables in terms of a well defined perturbation parameter. Canonical asymptotic far-field equations are obtained by eliminating secularly growing forced terms that come in at higher-order perturbation theory. The far-field equations are canonical in the sense that the same types of equations are derived from different physical systems. For example, the three-wave resonance equations are canonical because besides optics, they can also be obtained in fluid flows [8] and plasmas [9]. It must be noted that MMS may be used to derive the three-wave resonance equations that agree with SVEA. But here we are interested in deriving other canonical evolution equations not typically associated with quadratic nonlinear optical materials. For example, using MMS and proceeding to second-order perturbations, it has been recently shown [4,10,11] that the nonlinear Schrödinger equation is obtained for a single input wave propagating in a $\chi^{(2)}$ type material. Therefore, a quadratic nonlinear material may support intensity-dependent and self-modulation effects under appropriate conditions. This result is usually not associated with $\chi^{(2)}$ materials but is well known for $\chi^{(3)}$ materials, such as low attenuation glass optical fibers [12]. Here is one instance where MMS predicts a wider range of phenomena for $\chi^{(2)}$ materials than SVEA. The nonlinear Schrödinger equation arises in many other physical systems [13–15] and therefore is a canonical asymptotic evolution equation.

The present work is a generalization of the scalar one-dimensional problem presented in [10,11]. Here we apply two or three input electric fields to a quadratic nonlinear optical medium. This typically results in a system of two or three coupled partial differential equations derived from MMS. Several different problems are examined. At first we consider the effect of a constant dc field on the dielectric medium, which results in a nontraditional asymptotic wave-wave interaction process leading to a modified nonlinear Schrödinger equation as the fundamental optical frequency wave propagates through the medium. Then, we examine the effect of an applied slowly varying (microwave frequency) field interacting with the fundamental harmonic optical frequency wave and show that under appropriate conditions, a Schrödinger-type equation is coupled to a Korteweg–de Vries equation; another nontraditional asymptotic wave-wave mixing process for $\chi^{(2)}$ materials. Here, there is a long to short wave resonance if the phase velocity of the low-frequency field is close to the group velocity of the fundamental harmonic. We then consider the interaction of fundamental and second-harmonic optical waves. For relatively poor phase-matching conditions, coupled nonlinear Schrödinger equations are obtained that are analogous to the cross-phase-modulation equations of optical fibers [4,16]. In this nontraditional asymptotic regime the quadratic nonlinear medium behaves like a cubic nonlinear medium with intensity-dependent effects. For good phase matching equations similar to the three-wave resonance equations are obtained that include group velocity dispersion terms. This is the traditional asymptotic wave-wave process for $\chi^{(2)}$ materials. However, we provide interesting solutions to the wave processes.

The theory presented in this paper may be applied to specific $\chi^{(2)}$ materials. It should serve to initiate further experimental studies. This, in turn, may lead to different pulse-shaping applications of $\chi^{(2)}$ materials in optical communications. Most of these nontraditional effects may be observed in bulk materials or thin film waveguides that are tens of centimeters long. These material lengths are longer than millimeter lengths usually prescribed for harmonic-generation experiments. Pulse amplitude and widths should be chosen to minimize the effect of material absorption. Quadratic nonlinear materials in experiments should be chosen for low absorption and high damage thresholds. A suitable material may be potassium titaryl phosphate (KTP) as suggested in Refs. [10,11]. Numerical experimental conditions on nontraditional wave-wave processes will be presented in another paper.

So far, all the asymptotic equations are obtained by considering only scalar properties of the medium. Efficient phase matching is difficult to achieve for all input waves with the same polarizations propagating in the same direction. Finally, we take into consideration tensor properties of the medium (for type-II phase matching) when discussing interaction of waves of different polarizations for sum frequency generation.

Analytic solutions to the derived systems of evolution equations are obtained by reducing the complex valued coupled partial differential equations to systems of real valued ordinary differential equations by appropriately chosen phase parameters accompanying the slowly varying field amplitudes. Compatibility conditions are then given to ensure that the solitary-wave solutions are real. This technique enables the analytic description of solutions with nonzero walk-off parameters and nonzero dispersion terms. Other interesting solutions are given when dispersion terms are neglected and when one field is weak compared to the others as in nondepleted pump approximations.

## II. PROBLEM FORMULATION

We begin with the scalar medium model as presented in [10,11]. We allow two (or three) input waves of arbitrary shape, propagating in free space, to encounter a semi-infinite quadratic nonlinear material at $x = 0$. We assume normal incidence at the boundary and that propagation is in the $x$ direction, the electric fields are polarized in the $z$ direction, and their corresponding magnetic fields are in the negative $y$ direction. Most problems in this paper deal with electric fields polarized in the $z$ direction. However, in some problems, additional electric fields may be polarized in the $y$ direction, so that their associated magnetic fields are in the positive $z$ direction. The electric and magnetic fields are transverse to the direction of propagation, but they are tangential to the boundary. Therefore, for nonconstant fields, we impose boundary conditions that the tangential electric and magnetic fields be continuous across the boundary at $x = 0$. We also impose a radiation condition so that there are no incoming waves from infinity in the nonlinear material. We are given one slowly modulated field propagating at optical frequency $\omega$. For the two input wave problems, the other given field will be either a constant dc electric field applied to the medium or a slowly varying dc field (at microwave frequency) or a slowly modulated wave at the second harmonic $2\omega$. The resulting far-field evolution equations depend upon the type of input fields and also upon whether there exists a long to short wave resonance or harmonic to fundamental wave resonance.

The derived evolution equations depend not only on the quality of phase matching but also depend upon the pulse widths and the strength of the nonlinear coupling coefficients. In the three-input-wave problems, we assume that all frequencies lie in the optical range and we allow the medium to be anisotropic.

We consider nonlinear materials that can be described by an ensemble of identical classical and harmonic oscillators with small quadratic restoring forces and a resonant frequency $\omega_0$ far from $\omega$ (or $2\omega$). From the ensemble we obtain a macroscopic polarization $P$ that is coupled to the electric field $E$.

The above problem formulation translates into the following nondimensionalized equations. In free space the electric field $\tilde{E}$ satisfies Maxwell's wave equation with zero source terms

$$c^2 \frac{\partial^2}{\partial x^2}\, \tilde{E} - \frac{\partial^2}{\partial t^2}\, \tilde{E} = 0. \tag{1}$$

In the medium, the electric field $E$ and the polarization $P$ are coupled in the following manner:

$$\left(\frac{\partial^2}{\partial t^2} + \omega_0^2\right) P = fE - \epsilon^\alpha P^2 - \epsilon^2 \gamma \frac{\partial}{\partial t} P, \tag{2}$$

$$c^2 \frac{\partial^2}{\partial x^2} E - \frac{\partial^2}{\partial t^2} E = \frac{\partial^2}{\partial t^2} P, \tag{3}$$

with $\omega_0$ the dimensionless resonant frequency (equal to 1), $\sqrt{f}$ the dimensionless plasma frequency, $\gamma$ the dimensionless damping coefficient, and $c$ the dimensionless speed of light (equal to 1). The dimensionless perturbation parameter $\epsilon^\alpha$ is defined as

$$\epsilon^\alpha = \frac{dT^2 \epsilon_0 E_0^*}{Ne}. \tag{4}$$

It is a dimensionless ratio of the following dimensional variables: $d$ is the nonlinear restoring force coefficient, $T$ is a typical time scale $\omega_0^{-1}$ (inverse resonant frequency), $\epsilon_0$ is the permittivity of free space, $E_0^*$ is the typical field strength, $N$ is the electron density, and $e$ is the electron charge. Note that $E = E^*/E_0^*$, where $E^*$ is the given dimensional field.

The strength of $\epsilon^\alpha$ can be adjusted by choosing the typical time scale $T$ and field strength $E_0^*$. We allow $\alpha$ to be 1 or 2 depending on the problem; in Refs. [10,11], $\alpha$ was chosen to be 1.

Equations (2) and (3) model noncentrosymmetric crystals such as quartz and KTP. In general, the restoring force $d$ is a tensor of third rank, but for now we may assume that it is a scalar because we chose our crystal axis and polarizations so that only one component of $d, E$, and $P$ is required.

It is convenient to operate on Eq. (3) by $\left(\frac{\partial^2}{\partial t^2} + \omega_0^2\right)$ and replace Eq. (3) by the fourth-order equation

$$\left(\frac{\partial^2}{\partial t^2} + \omega_0^2\right) \left(c^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}\right) E - f \frac{\partial^2}{\partial t^2} E = -\epsilon^\alpha \frac{\partial^2}{\partial t^2} P^2 - \epsilon^2 \gamma \frac{\partial^3}{\partial t^3} P. \quad (5)$$

Equation (5) is important because the dispersion properties of the medium are included and the polarization terms act as known forced terms that come in at higher orders. It is also important that all secularity conditions on the shortest and fastest scales for MMS are determined from the $\epsilon$ independent left-hand side. Equation (5) describes the medium accurately as long as we remain in the optical frequency range between the infrared and ultraviolet resonance bands. If we cross the infrared resonance band, Eq. (2) must be supplemented by additional oscillators at each resonance band. The refractive index will increase across the resonance band when the frequency is lowered (as explained in [11]). To accurately describe the linear refractive index, we must replace Eq. (2) by

$$P_j = \int_{-\infty}^{\infty} \chi_{jk}^{(1)}(t-\tau) E_k(\tau) d\tau + \epsilon^\alpha \int_{-\infty}^{\infty} d\tau_1 d\tau_2 \chi_{jk\ell}^{(2)}(t-\tau_1, t-\tau_2) E_k(\tau_1) E_\ell(\tau_2). \quad (6)$$

The subscripts $jk\ell$ refer to the field components [17]. These subscripts were suppressed because we assumed $j = k = \ell = z$. Now we must solve Eqs. (3) and (6) in conjunction. If we substitute (6) into (3) we have an integro-differential equation. It is interesting to note that Eq. (6) may be derived from Eq. (2) by applying Fourier transforms to perturbation theory. The linear and quadratic susceptibilities $\chi_{jk}^{(1)}, \chi_{jk\ell}^{(2)}$ are specified and restricted by Eq. (2) to the optical frequency range. However, we allow $\chi^{(1)}$ and $\chi^{(2)}$ to vary as needed to describe the medium. The integro-differential equation formulation described by Eqs. (6) and (3) is equivalent to the fourth-order differential equation given by Eq. (5). They are identical in the optical frequency range. Thus we may derive our asymptotic far-field equations from either formulation, depending on our needs.

We also have the boundary conditions

$$\tilde{E}(0,t) = E(0,t) \ , \ \tilde{H}(0,t) = H(0,t), \quad (7)$$

with $\tilde{H}, H$ being, respectively, the magnetic fields in free space and the medium. Now we shall apply MMS to the above system of equations.

MMS is a singular perturbation technique that seeks uniform or sinusoidal solutions after eliminating artificial secularly growing terms by expanding both dependent and independent variables in a perturbation series [5]. We assume that the independent variables $(x,t)$ become the set of $2n$ independent variables $(x_0, x_1, x_2, x_3, \ldots, x_n; t_0, t_1, t_2, t_3, \ldots, t_n)$ with $(x_0 = x, t_0 = t)$ and $x_i = \epsilon^i x_0,\ t_i = \epsilon^i t$ for $i = 1, 2, \ldots, n$.

Depending on our problem, $(x,t)$ is expanded to either three or four spatial and time scales. The $(x_0, t_0)$ variables are, respectively, the shortest distance and fastest time scales. The other independent variables are longer distance or slower time scales. The derivative operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$ are expanded as

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \epsilon \frac{\partial}{\partial x_1} + \epsilon^2 \frac{\partial}{\partial x_2} + \epsilon^3 \frac{\partial}{\partial x_3} + \cdots + \epsilon^n \frac{\partial}{\partial x_n}, \quad (8)$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \epsilon \frac{\partial}{\partial t_1} + \epsilon^2 \frac{\partial}{\partial t_2} + \epsilon^3 \frac{\partial}{\partial t_3} + \cdots + \epsilon^n \frac{\partial}{\partial t_n}. \quad (9)$$

We assume further that the $P, \tilde{E}, E$ fields are regarded as functions of the $2n$ independent variables and have an asymptotic representation of the form

$$\begin{aligned} P(x_0, x_1, \ldots, x_n; t_0, t_1, \ldots t_n) &= P_0(x_0, x_1, \ldots, x_n; t_0, t_1, \ldots t_n) \\ &\quad + \epsilon P_1(x_0, x_1, \ldots, x_n; t_0, t_1, \ldots t_n) \\ &\quad + \epsilon^2 P_2(x_0, x_1, \ldots x_n; t_0, t_1, \ldots t_n) + \cdots . \end{aligned} \quad (10)$$

Similar expansions exist for $E, \tilde{E}, H, \tilde{H}$. We substitute expansions (8)–(10) into Eqs. (1), (2), (5), and (7), or (1), (3), (6), and (7), depending on the problem, and collect terms of the same order of $\epsilon$. At each perturbation level, we obtain equations for each harmonic component of the electric field. At higher perturbation orders, forced resonant or secular terms for each harmonic component are set to zero. These secularity conditions produce the asymptotic far-field equations that depend on the slow time and long distance scales. We now summarize some results.

For the $O(1)$ perturbation level boundary value problem we assume the electric field in free space has the following form for each harmonic component:

$$\begin{aligned} \tilde{E} = &\ \tilde{a}_m(t_1 - x_1/c) \exp[im\omega(x_0/c - t_0)] \\ &+ \tilde{b}_m(t_1 + x_1/c) \exp[-im\omega(x_0/c + t_0)] + \text{c.c.}, \end{aligned} \quad (11)$$

where $\tilde{a}_m$ is the given slowly modulated envelope that is dependent on the $(t_1, x_1)$ scales. The reflected wave envelope $\tilde{b}_m$ is unknown and has to be determined from the $O(1)$ boundary conditions. The integer $m$ indicates the harmonic component. For $m = 0, 1, 2$, we have, respectively, the slowly varying dc field, the fundamental harmonic, and the second harmonic.

In the medium, we assume that the $O(1)$ electric field has the following form for $m = 1, 2$ harmonic components:

$$E_0 = a_m(x_1, x_2, x_3; t_1) \times \exp\{im\omega[x_0 k(m\omega)/(m\omega) - t_0]\} + \text{c.c.} \qquad (12)$$

For slowly varying dc field ($m = 0$) we assume further that

$$E_0 = A(t_1 - x_1/v_0; x_2, x_3) + \text{c.c.} \ , \qquad (13)$$

with $A = a_0$ and $v_0 = v$, where $v_0 = \tilde{\omega}/k(\tilde{\omega})$ is the given phase velocity associated with the centroid $\tilde{\omega}$ of the wave packet. Note that $\tilde{\omega} \ll \omega$ and $v$ is the phase velocity due to an $O(1)$ low-frequency field. Here the $a_m$ for $m = 0, 1, 2$ are the unknown transmitted wave envelopes. Substituting the assumed form for the $O(1)$ electric field for the $m = 1, 2$ components shown by Eq. (12) into Eq. (5) for the $O(1)$ perturbation level, we obtain the dispersion relations

$$D(k(m\omega), m\omega) = c^2 k^2(m\omega)(m^2\omega^2 - \omega_0^2) - m^4\omega^4 + m^2(\omega_0^2 + f)\omega^2 = 0. \qquad (14)$$

The wave numbers $k(m\omega)$ may be expressed in terms of their applied frequencies. The reflected and transmitted wave envelopes can be expressed in terms of the given incident field by applying the boundary conditions (7) for each harmonic ($m = 0, 1, 2$):

$$\tilde{b}_m(t_1) = \frac{v_m - c}{v_m + c}\, \tilde{a}_m(t_1) \ , \qquad (15)$$

$$a_m(t_1) = \frac{2v_m}{v_m + c}\, \tilde{a}_m(t_1) \ , \qquad (16)$$

with $v_m = m\omega/k(m\omega)$ the phase velocity for $m$ =1,2, whereas $v = \tilde{\omega}/k(\tilde{\omega})$ for $m = 0$ denoting the $O(1)$ low-frequency field.

The above $O(1)$ results appear to be the standard results from linear theory; however, Eqs. (15) and (16) have been generalized as in [4,10,11] to include any pulse shape that can be parametrized by the $t_1$ time scale. Similar results are obtained for the three-wave problem.

In this paper we are primarily interested in how the $O(1)$ fields are slowly modulated and seek the resulting envelope or far-field equations obtained from secularity conditions that come in at higher perturbation orders. We are not interested in the $O(\epsilon)$ electric fields generated by the bounded source terms and will not present the $O(\epsilon)$ boundary value problems since they have already been dealt with in [4,10,11].

The $O(\epsilon)$ secularity conditions imply that the $O(1)$ fields obey first-order partial differential equations on the $(x_1, t_1)$ scales for the optical frequency waves. The $O(\epsilon^2)$ secularity conditions imply that $O(1)$ fields obey second-order partial differential equations. Usually there exist second-order derivatives on the $(x_1, t_1)$ scales and first-order partial derivatives on the $(x_2, t_2)$ scales. The physically interesting canonical far-field equations in the optical frequency range appear at this perturbation level. They take the appearance of various nonlinear Schrödinger equations. At $O(\epsilon^3)$, we find that there may be small corrections to the effective nonlinear refractive index as shown in [11]. These corrections are due to a self-induced $O(\epsilon)$ rectified field propagating at phase velocity $V$. This is not to be confused with the given $O(1)$ low-frequency field. For the given applied slowly varying $O(1)$ low-frequency field dependent on the $(x_1, t_1)$ scales, we find that the secularity conditions at $O(\epsilon^2)$ result in a dispersionless wave equation. This wave propagates at phase velocity $v$. Now, we must proceed to the $O(\epsilon^4)$ perturbation theory. The secularity conditions imply that the partial differential equation is of third order on the $(x_1, t_1)$ scales and first order on the $(x_3, t_3)$ scales. We obtain a Korteweg–de Vries equation that may be coupled to a Schrödinger-type equation for the optical field. For the sum frequency problem, second-order perturbation theory produces the main results, which are analogous to the fundamental to second-harmonic wave interaction problem. However, the resulting three interacting nonlinear Schrödinger equations will change form, depending upon the phase-matching efficiency. These evolution equations are dominated respectively by quadratic or cubic nonlinearities if the phase matching is efficient or inefficient.

In order to avoid rewriting redundant equations, the derived canonical evolution equations will be expressed in dimensional or physical variables. That is, the resulting dimensionless far-field evolution equations obtained from MMS dependent on the slow time and distance scales will be converted back to the original time and distance variables and the subsequent physical variables. (This is also useful for experimental and numerical results.) If it is necessary, one may infer the multiple scale dependence by the order of the derivative as described in the preceding paragraph. For details on how to apply MMS theory and convert from dimensionless to dimensional variables for the optics signaling problem presented, the reader should consult [4,11]. A general treatment of MMS is given in [5]. As mentioned before, the two- and three-input wave problem is a generalization of the single-input wave problem given in [10,11]. We will at first present the most robust coupled equations derived from MMS and then discuss various analytic solutions or analytic approximations. We shall also discuss relevant simpler systems of equations by neglecting certain terms of the fully derived equations. Numerical results will be presented in a separate paper.

Here we discuss the notation used. The field variables $(a, A, b)$ denote, respectively, the slowly varying amplitudes of the fundamental harmonic, the constant or low-frequency field, and the second harmonic for the two input waves. Similarly $(a, b, u)$ denote slowly varying field amplitudes at frequencies $(\omega_1, \omega_2, \omega_3)$ for the three-input-wave problem. The reduced time coordinate $s = t - k'(\omega)x$ is consistent and has the same meaning throughout the manuscript. Corrections to $s$ are denoted by variables such as $\tilde{s}$, $s_1$, etc.

## III. THE FUNDAMENTAL HARMONIC AND CONSTANT FIELDS

The fundamental harmonic field with slowly modulated amplitude $a$ is incident upon a quadratic nonlinear medium immersed in a constant electric field $A$. The polarizations are all in the same direction. The total applied electric field in the medium is expressed as

$$E = a\, e^{i\theta} + a^* e^{-i\theta} + A, \tag{17}$$

with $\theta = k(\omega)x - \omega t$ dependent on the $(x_0, t_0)$ scales and $a$ dependent on the slow scales. Here Eq. (17) is written in dimensional form. We assume that the parameter $\alpha$ shown in Eqs. (2) and (5) is one. The following $O(\epsilon)$ evolution equation is obtained from MMS for the slowly modulated field $a$,

$$\frac{\partial a}{\partial x} + \frac{\partial k}{\partial \omega}\frac{\partial a}{\partial t} + iqAa = 0, \tag{18}$$

with

$$q = \frac{d\epsilon_0 \omega_p^4 \omega^2}{k(\omega) N e c^2 \omega_0^2 (\omega_0^2 - \omega^2)^2}\ . \tag{19}$$

The plasma frequency is $\omega_p$ and the other variables have already been defined in Sec. II. The solution to Eq. (18) is obtained by the method of characteristics:

$$a = \mathcal{E}(s)\, e^{-iqAx}\ , \tag{20}$$

with the reduced time $s = t - k'(\omega)x$. The envelope $\mathcal{E}$ is constant along $s$. We may interpret the phase factor $qA$ as an $O(\epsilon)$ correction to the wave number $k(\omega)$. That is, we have a new effective wave number $\tilde{k}(\omega) = k(\omega) - qA$ dependent upon the applied constant field $A$.

At second-order perturbation theory, we find the dominant evolution equation for the slowly modulated field $E$ given in Eq. (20),

$$i\frac{\partial}{\partial x}\mathcal{E} + i\left(\frac{\partial k}{\partial \omega} - \frac{\partial q}{\partial \omega}A\right)\frac{\partial \mathcal{E}}{\partial t} - \frac{1}{2}k''(\omega)\frac{\partial^2}{\partial t^2}\mathcal{E}$$

$$-\beta(\omega)[1+g(\omega)]|\mathcal{E}|^2\mathcal{E} + \beta_1 A^2 \mathcal{E} = -i\frac{\sigma(\omega)}{2}\,\mathcal{E}. \tag{21}$$

The following coefficients are defined as

$$\beta(\omega) = \frac{4}{3}\,\frac{[d^{(2\omega)}]^2(\omega_0^2 - 4\omega^2)(\omega_0^2 - 6\omega^2)}{c^2\omega_p^2\omega_0^2 k(\omega)}\ , \tag{22}$$

$$g(\omega) = \frac{6\omega^2\omega_p^2}{(\omega_0^2 - 6\omega^2)\omega_0^2 c^2\left[\frac{1}{V^2} - \frac{1}{v_g^2(\omega)}\right]}\ , \tag{23}$$

$$\beta_1 = q^2\left[\frac{kc^2(3\omega_0^2 - \omega^2)(\omega_0^2 - \omega^2)}{\omega_p^2\omega_0^2\omega^2} - \frac{1}{2k}\right]\ , \tag{24}$$

the absorption coefficient $\sigma(\omega)$ is

$$\sigma(\omega) = \frac{\omega^3\Gamma\omega_p^2}{c^2(\omega_0^2 - \omega^2)^2 k(\omega)}, \tag{25}$$

where $\Gamma = \epsilon^2\gamma$, and the second-harmonic generation coefficient $d^{(2\omega)}$ is

$$d^{(2\omega)} = \frac{dNe^3}{2m^2(\omega_0^2 - \omega^2)^2(\omega_0^2 - 4\omega^2)\epsilon_0}\ . \tag{26}$$

Note that $m$ is the electron mass, $v_g(\omega) = k'(\omega)^{-1}$, and $V$ is the phase velocity of a rectified $O(\epsilon)$ field.

Equation (21) is the nonlinear Schrödinger equation that includes the effects of a constant field $A$. There is an $O(\epsilon)$ correction to the group velocity that is linear in $A$ and a correction to the phase of $\mathcal{E}$ that is quadratic in $A$. Note that if $A$ is set to zero, we recover the nonlinear Schrödinger equation for quadratic nonlinear dispersive optical materials that was derived in Refs. [4,10,11]. The one-soliton solution to Eq. (21) for $\sigma = g = 0$ with given peak amplitude $\mathcal{E}_0$ is

$$\mathcal{E} = \mathcal{E}_0\,\mathrm{sech}\left(\left|\frac{\beta}{k''(\omega)}\right|^{1/2}\mathcal{E}_0\,\tilde{s}\right)e^{-i(\beta\mathcal{E}_0^2/2 - \beta_1 A^2)x} \tag{27}$$

with characteristic

$$\tilde{s} = t - \left(\frac{\partial k}{\partial \omega} - \frac{\partial q}{\partial \omega}A\right)x. \tag{28}$$

For $A = 0$, Eqs. (27) and (28) reduce to the one-soliton solution given in the above-mentioned references. Note that at $O(\epsilon^2)$ for the total fundamental harmonic field, we replace $\mathcal{E}(s)$ with characteristic $s$ in Eq. (20) by the field given in Eq. (27) with characteristic $\tilde{s}$ given by Eq. (28). The application of a constant electric field does not change the relationship between the soliton peak amplitude and pulse width or soliton period. Thus, for the the same experimental conditions for KTP, as shown in Refs. [10,11], when a peak field amplitude is $5.0 \times 10^7$ V/m and the corresponding pulse width is 0.1 psec, then these imply that the soliton period is still 8.9 cm. However, the soliton does propagate at a different group velocity.

If we analyzed a medium with intrinsic cubic nonlinearity immersed in a constant field instead of a quadratic nonlinear material, we would have to replace the $\epsilon P^2$ terms in Eqs. (2) and (5) by $\epsilon^2 P^3$. Using the same MMS procedure as before, we find that at $O(\epsilon)$, $q = 0$ in Eqs. (18) and (20). There is no correction to the wave number. At second-order perturbation theory, there is no correction to the group velocity and hence $\tilde{s} = s$. The one-soliton solution Eq. (27) only has the phase delay $\beta_1 A^2$. Of course, $\beta_1$ is different for a cubic material and is different from Eq. (24). Instead $\beta_1 = kn_2[(\omega_0^2 - \omega^2)/\omega_0^2]^2/n$.

## IV. THE FUNDAMENTAL HARMONIC AND SLOWLY VARYING dc FIELD

The same fields are applied as in the previous problem but we allow the dc field $A$ to be slowly varying on the $(t_1, x_1)$ scales. We assume also that $A$ is so slowly modulated that its group velocity is essentially the phase velocity because the pulse is essentially monochromatic. To derive a canonical system of equations that balances dispersion with nonlinearity, we consider the case of short to long wavelength resonance whereby the phase velocity $v$ of $A$ is nearly equal to the group velocity $v_g(\omega)$ of the fundamental field $a$. In this asymptotic regime, we also assume the peak pulse amplitude is slightly weaker so that $\epsilon^\alpha$ defined by Eq. (4) becomes $\epsilon^2$ where $\alpha = 2$, instead of one as in the previous section. This new choice of $\alpha$ enables quadratic nonlinearities (instead of cubic) to be balanced with dispersion. Note that from Eq. (4) we could keep the peak power the same as before but use narrower pulse widths to change $\alpha$ from 1 to 2. The change in peak pulse amplitude to its respective pulse width results in different asymptotic equations. Using MMS, we proceed to $O(\epsilon^2)$ for the fundamental field $a$ whereas for the dc field $A$, we begin at $O(\epsilon^2)$ and continue to the $O(\epsilon^4)$ perturbation level.

We find that at $O(\epsilon)$, the optical frequency wave obeys the first-order equation

$$\frac{\partial a}{\partial x} + \frac{1}{v_g(\omega)} \frac{\partial}{\partial t} a = 0. \tag{29}$$

Equation (29) shows that the envelope $a$ propagates with its group velocity $v_g(\omega)$ and is constant along the characteristic $s = t - k'(\omega)x$. However, at second-order perturbation theory, we find that $a$ instead obeys

$$i\left(\frac{\partial}{\partial x} + k'(\omega)\frac{\partial}{\partial t}\right) a - \frac{1}{2}k''(\omega)\frac{\partial^2}{\partial t^2}a - qAa + i\frac{\sigma(\omega)}{2}a = 0. \tag{30}$$

The coefficients $\sigma(\omega)$ and $q$ were previously defined in Eqs. (25) and (19) in the optical frequency regime. In general,

$$q = \frac{\omega^2 \chi^{(2)}(\omega; 0, \omega)}{2c^2 k(\omega)}$$

if we use the integro-differential equation obtained from Eqs. (3) and (6). Physically, Eq. (30) describes an $O(\epsilon)$ deviation from the characteristic solution given in Eq. (29). Here $a$ is no longer kept constant along characteristic $s$. Equation (30) is a nonlinear Schrödinger equation with a quadratic nonlinearity that couples the optical field to the slowly varying field.

Applying MMS to the dc field, we find that at $O(\epsilon^2)$ and $O(\epsilon^3)$ $A$ obeys the wave equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) A = 0 , \tag{31}$$

with $v = \tilde{\omega}/k(\tilde{\omega})$. We see that $A$ propagates with the phase velocity $v$. At $O(\epsilon^4)$ perturbation theory, we find that now $A$ is described by

$$\left(\frac{\partial}{\partial x} + \frac{1}{v}\frac{\partial}{\partial t}\right) A - b_1 \frac{\partial^3}{\partial t^3} A + b_2 \frac{\partial}{\partial t} A^2 + b_3 \frac{\partial}{\partial t}|a|^2 = 0. \tag{32}$$

If we approach the infrared resonance, but do not cross it, we may derive Eq. (32) from Eq. (5), but if we do cross the resonance band, we must start with Eqs. (3) and (6). This changes the coefficients $b_1, b_2, b_3$. We will at first give the general form of the coefficients and then the restricted form derived from Eq. (5):

$$b_1 = \frac{n''(\tilde{\omega})}{2c} = \frac{v}{2c^2\omega_0^2}\left(\frac{c^2}{v^2} - 1\right) ,$$

$$b_2 = \frac{v}{2c^2}\,\chi^{(2)}(0; \tilde{\omega}, -\tilde{\omega}) = -\frac{qvk}{\omega^2}\frac{(\omega_0^2 - \omega^2)^2}{2\omega_0^2} ,$$

$$b_3 = \frac{v}{2c^2}\chi^{(2)}(0; \omega, -\omega) = -\frac{vkq}{\omega^2} .$$

Equation (32) may be interpreted physically as an $O(\epsilon^2)$ correction to the characteristic solution of Eq. (31). At this level, $A$ is no longer constant along the characteristic $t - x/v$. The velocity of propagation of $A$ is no longer $v$, and will depend on the amplitude of $A$. Equation (32) is a Korteweg–de Vries (KDV) equation for $A$ with a source term obtained from the optical field. It is coupled to Eq. (30) and they must be solved simultaneously. The system (30) and (32) is analogous to the system derived for electron plasma waves interacting with ion acoustic waves in [18] and long waves interacting with short waves in capillary-gravity waves in [19]. It must be stated that in using MMS, the Boussinesq equation analogy may be derived instead of the KDV-like equation (32). The Boussinesq equation reduces to Eq. (32) asymptotically for waves traveling in one direction.

We seek some special solutions to Eqs. (30) and (32). We are interested in solitary-wave solutions. First, we note that if $a = 0$, we have the KDV equation. For $b_1 > 0$ and $b_2 < 0$, we find that the one-soliton solution to Eq. (32) for the imposed conditions is

$$A = A_0 \mathrm{sech}^2\left[\sqrt{\frac{A_0 b_2}{6\, b_1}}\left(t - \frac{1}{v}x \pm \frac{2}{3}b_2 A_0 x\right)\right] , \tag{33}$$

where the initial amplitude $A_0 > 0$ at $x = 0$. For these conditions we use the plus sign for the $A_0 x$ term. The one-soliton solution also exists if $b_1 < 0$, $b_2 > 0$, and $A > 0$, but we use the minus sign [in Eq. (33)].

The coupling of Eq. (30) to (32) is strongest if the group velocity $v_g(\omega)$ is very close to the phase velocity $v$. We now assume both fields $a, A$ are nonzero, $\sigma(\omega) = 0$, and $v_g(\omega) - v \sim \epsilon$. We present a solitary-wave solution from [18,19]. We assume a reduced time $\tau = t - \lambda x$ and reduce Eqs. (30) and (32) to ordinary differential equations in the manner done in [18,19]. For the case $q > 0, k''(\omega) < 0, b_1 < 0, b_2 > 0$, and $b_3 > 0$, it can be shown that

$$a = a_0\delta^2 \mathrm{sech}\delta\tau \tanh\delta\tau e^{i\phi\tau},$$
$$A = -\frac{3k''(\omega)}{q}\delta^2 \mathrm{sech}^2\delta\tau \ , \quad (34)$$

with

$$a_0^2 = \frac{18k''(\omega)}{q\ b_3}\left(\frac{b_2\ k''(\omega)}{2q} + b_1\right)$$

and

$$\delta^2 = \frac{q\ A_0}{k''(\omega)}\sqrt{\frac{b_2}{2b_3}} - \left(\frac{1}{v_g(\omega)} - \frac{1}{v}\right)^2 / [k''(\omega)]^2 .$$

Here $A_0$ is an integration constant that must be positive enough so that $\delta^2 > 0$. The constant $A_0$ is the initial amplitude of $A$. The inverse phase velocity $\lambda$ is

$$\lambda = \frac{1}{v} - \delta^2\left(\frac{3k''(\omega)}{q}b_2 + 2b_1\right) + \sqrt{2b_3b_2}\ A_0 \ .$$

Finally, the phase factor $\phi$ in Eq. (34) is given

$$\phi = -\left(\frac{1}{v} - \lambda\right) / k''(\omega) \ .$$

The same sort of solitary-wave solution may be obtained for parameters of negative sign. That is, for $q < 0$, $b_2 < 0$, $b_3 < 0$, $b_1 > 0$, and $k''(\omega) > 0$, we obtain Eq. (34), but $\phi$ is negative and the inverse phase velocity $\lambda$ has all positive signs.

Another solution similar in form inspired by but not discussed in [18,19] may be obtained if $v_g(\omega) \neq v$ and $v_g(\omega) - v \sim 1$. We reduce the coupled system (30) and (32) to a pair of real second-order ordinary differential equations by adjusting the phase of the optical field $a$ to include two parameters $r, h$ instead of $\phi$. This change gives an additional degree of freedom to ensure real solutions. We assume that $a = \tilde{a}(s_i)e^{i(rs+hx)}$ and $A = A(s_i)$ with $s = t - k'(\omega)x$ and $s_i = s - \bar{\lambda}x$. We choose $r = \bar{\lambda}/k''(\omega)$ and then substitute the assumed forms into Eqs. (30) and (32) to obtain real coupled ordinary differential equations with independent variable $s_i$:

$$\tilde{a}'' + \left(r^2 + \frac{2h}{k''(\omega)}\right)\tilde{a} = -\frac{2qA\tilde{a}}{k''(\omega)}, \quad (35)$$

$$A'' + \left(\frac{\bar{\lambda} + k'(\omega) - \frac{1}{v}}{b_1}\right)A = \frac{b_2}{b_1}A^2 + \frac{b_3}{b_1}|\tilde{a}|^2. \quad (36)$$

A solution $\tilde{a}(s_i) = a_0 \mathrm{sech}\ \delta s_i \tanh\ \delta s_i$ and $A = A_0 \mathrm{sech}^2\delta s_i$ exists provided the following compatibility conditions hold:

$$\delta^2 = \left(r^2 + \frac{2h}{k''(\omega)}\right)$$
$$= \frac{q\ A_0}{3k''(\omega)}$$
$$= \left(\frac{b_3}{b_1}\frac{a_0^2}{A_0} - \frac{b_2}{6b_1}A_0\right)$$
$$= \left[\frac{(\bar{\lambda} + \frac{1}{v_g(\omega)} - \frac{1}{v})}{4\ b_1} + \frac{b_3\ a_0^2}{4\ b_1\ A_0}\right] .$$

The parameters $\bar{\lambda}, \delta^2, a_0^2, A_0^2$ must be real and positive. The parameter $h$ must be real but may also be negative if needed. We choose $h$ to make sure that there are no contradictions. The parameters $\bar{\lambda}, \delta^2, a_0^2, h$ may be expressed in terms of $A_0$, which is arbitrary. Also note that the solutions to Eqs. (35) and (36) also hold if $v_g(\omega) = v$. Note that by choosing the phase factor for field $a$ to depend on $rs + hx$ rather than the reduced times $\bar{s}$ or $\tau$, we are able to obtain solutions where $v_g(\omega) \neq v$. We chose parameters $r, h$ so that Eq. (35) is real and the resulting solutions are real. This procedure to reduce systems of complex valued partial differential equations to real ordinary differential equations by appropriate choices of phase parameters is used throughout the text. The phase factor compensates so that the envelopes of Eqs. (35) and (36) propagate with inverse velocity $\bar{\lambda}$. We now discuss some other types of solutions for the case when $v_g(\omega) - v \sim 1$.

Under appropriate conditions, the problem simplifies somewhat if there is no short to long wave resonance, or $v_g(\omega) - v \sim 1$. We may decouple Eqs. (30) and (32) using the fact that Eq. (32) is an $O(\epsilon^2)$ correction to the characteristic solution of Eq. (31). In this regime, we neglect the change in $A$ and assume that it is known and propagates with phase velocity $v$. Equation (30) alone is sufficient to describe how the optical field $a$ is influenced by the low-frequency field acting as a known source term thus making Eq. (32) linear.

Assuming $A = \frac{k''(\omega)}{q}\ \mathrm{sech}^2(s_0)$, with $s_0 = t - x/v$, $\sigma \neq 0, k''(\omega) > 0$, and $q > 0$, we find the corresponding harmonic field is

$$a = a_0\ e^{\frac{-\sigma(\omega)}{2}x}\ e^{-ihx}\ e^{is_0q/k''(\omega)} \mathrm{sech}\ s_0, \quad (37)$$

with initial amplitude $a_0$ and $h = \frac{k''}{2} + \frac{1}{2k''}\left(\frac{1}{v_g(\omega)} - \frac{1}{v}\right)^2$. It is interesting to note here that the solution (37) is expressed in terms of the characteristic of the low-frequency field. In addition, if we increase $A$ by a factor of 3, we have to replace $\mathrm{sech}(s_0)$ by $\mathrm{sech}(s_0)\tanh(s_0)$. Also, if we neglect damping and group velocity dispersion by setting $\sigma(\omega) = k''(\omega) = 0$ in Eq. (30), we are left with a first-order differential equation that can be solved using characteristics. That is, Eq. (18) is obtained, but $A$ is not constant (as assumed before). If $A = \frac{\ell_0}{q}\ \mathrm{sech}^2 s_0$, with $\ell_0 = \left(k'(\omega) - \frac{1}{v}\right)$, we find that

$$a = \mathcal{E}(s)\ e^{-i\tanh s_0} = \mathcal{E}(s)\exp\left(-iq/\ell_0\int_0^{s_0} Ads'\right) \quad (38)$$

with initial pulse $\mathcal{E}$ propagating along the characteristic $s = t - \frac{1}{v_g(\omega)}x$. In this regime, the optical wave $a$ propagating in a quadratic nonlinear material may be used in an interferometric configuration to detect the presence of slowly varying (infrared or microwave) fields. The presence of the microwave $A$ causes a phase change in the optical field envelope $a$.

We have just shown how a $\mathrm{sech}^2(t)$ pulse for the dc field at the boundary interacts with the harmonic field $a$ in the various asymptotic regimes. Short to long wave resonance involves nonlinear dynamics, whereas away from resonance, linear dynamics may be dominant under the appropriate conditions. If $A$ is constant, under appropriate conditions, the dispersionless Eq. (18) is obtained but the dispersive, cubic, nonlinear Schrödinger equation (21) cannot be rederived under this regime. This is because field intensities and pulse widths were chosen to balance dispersion with quadratic nonlinearities resulting in Eq. (30). For constant $A$ and zero damping, the solution to Eq. (30) consists of a phase factor $\exp(-iqAx)$ multiplying a pulse profile $\exp\{is^2/[2k''(\omega)x)]\}/\sqrt{x}$.

## V. THE INTERACTION OF FUNDAMENTAL AND SECOND HARMONIC WAVES

Here we apply two electric fields respectively propagating at optical carrier frequencies $\omega$ and $2\omega$ in the nonlinear quadratic medium. The $O(1)$ field $E$ in the medium is of the form

$$E = \frac{a}{\sqrt{k(\omega)}}\, e^{i\theta} + \frac{\sqrt{2}\, b}{\sqrt{k(2\omega)}}\, e^{i[k(2\omega)x - 2\omega t]} + \text{c.c.} \qquad (39)$$

The $O(\epsilon)$ perturbation level shows that each slowly varying envelope propagates with the respective group velocity associated with its harmonic and each envelope obeys equations analogous to Eq. (29). The envelopes do not interact until $O(\epsilon^2)$ perturbation theory. There are two asymptotic regimes at this level that depend on the efficiency of the phase matching. There is a system of equations for efficient $O(\epsilon)$ phase matching that is different from inefficient $O(1)$ phase matching. When we consider $O(\epsilon)$ phase matching, we assume a certain relationship between the pulse period and pulse intensity. The pulse period is $O(\epsilon)$ and the strength of the nonlinear terms is $O(\epsilon^2)$ so that $\alpha = 2$ in Eq. (4). Here $\alpha$ is chosen to balance dispersion with quadratic nonlinearity. For $O(1)$ phase mismatch, we assume the pulse period is the same as before but the strength of the nonlinear term is $O(\epsilon)$, which means $\alpha = 1$ in Eq. (4). This enables cubic nonlinearities to balance with dispersion. The phase mismatch between wave numbers is given as $\Delta k = k(2\omega) - 2k(\omega)$. We first consider the case when $\Delta k \sim \epsilon$.

## VI. $O(\epsilon)$ PHASE-MISMATCH EQUATIONS

MMS second-order perturbation theory with efficient phase matching results in the following system [4] for the fundamental and second envelope $(a, b)$:

$$i\left(\frac{\partial}{\partial x} + \frac{1}{v_g(\omega)}\frac{\partial}{\partial t}\right) a - \frac{k''(\omega)}{2}\,\frac{\partial^2}{\partial\, t^2}a - \kappa e^{i\Delta k x}\, a^* b = -i\frac{\sigma(\omega)}{2}a, \qquad (40)$$

$$i\left(\frac{\partial}{\partial x} + \frac{1}{v_g(2\omega)}\frac{\partial}{\partial t}\right) b - \frac{k''(2\omega)}{2}\,\frac{\partial^2}{\partial\, t^2}b - \kappa e^{-i\Delta k x} a^2 = -i\frac{\sigma(2\omega)}{2}b, \qquad (41)$$

with $\kappa = 2^{3/2}\, \omega^2\, d^{(2\omega)}/[c^2 k(\omega)\sqrt{k(2\omega)}]$. Equations similar to Eqs. (40) and (41) for quadratic optical materials are well known and were obtained by Akhmanov *et al.* [20] using a different perturbation method. These equations may be considered degenerate three-wave-mixing equations that include group velocity dispersion and pulse walk-off effects. These equations are coupled nonlinear Schrödinger equations with quadratic nonlinearities. The above equations are probably best studied using numerical methods such as the split-step Fourier method, but we will attempt to obtain solitary-wave solutions analytically using various approximations. In all cases, we neglect damping so that $\sigma(\omega) = \sigma(2\omega) = 0$. At first, we review several solutions that neglect group velocity dispersion terms.

For the time-independent problem, Eqs. (40) and (41) reduce to

$$\frac{\partial}{\partial x}a = -i\kappa\, e^{i\Delta k x}\, a^*\, b, \qquad (42)$$

$$\frac{\partial}{\partial x}b = -i\kappa e^{-i\Delta k x}\, a^2. \qquad (43)$$

This system was studied by Armstrong *et al.* [21]. Recently, De Salvo *et al.* [22] created new interest in this problem when they experimentally observed intensity dependent effects in quadratic materials due to what they termed "cascading effects." They were able to obtain a complex Duffing equation with cubic nonlinearity by differentiating Eq. (42) and utilizing the conservation law $|a|^2 + |b|^2 = A_0^2$ with $A_0$ constant:

$$\frac{\partial^2}{\partial x^2}\, a = i\Delta k\, \frac{\partial a}{\partial x} + \kappa^2\, A_0^2 a - 2\kappa^2 |a|^2 a\;. \qquad (44)$$

Upon further examination, with $a = B\, e^{iW}$ and $(B, W)$ real, we find that Eq. (44) may be reduced to the set of equations

$$\frac{\partial^2 B}{\partial x^2} = \left[\kappa^2 A_0^2 - \frac{(\Delta k)^2}{4}\right]\, B - 2\kappa^2 B^3 + \frac{A_1}{B^3} \qquad (45)$$

and

$$\frac{\partial W}{\partial x} = \frac{A_1}{B^2} + \frac{\Delta k}{2}, \qquad (46)$$

where $A_1$ is a second integration constant. Equation (45) can be integrated once to give

$$\left(\frac{\partial B}{\partial x}\right)^2 = \left[\kappa^2 A_0^2 - \frac{(\Delta k)^2}{4}\right]\, B^2 - \kappa^2\, B^4 - \frac{A_1}{B^2} + A_2, \qquad (47)$$

with $A_2$ another constant. Equations (46) and (47) are consistent with and contained within the work of Armstrong *et al.* [21] and may be solved by elliptic integrals. Both Armstong *et al.* [21] and De Salvo *et al.* [22] utilize the same conservation law in their respective but equivalent approaches. It is interesting to note that for $A_1 = A_2 = 0$, the fields satisfy a signaling problem with

$$a = \tilde{A}_0 \text{sech}\,(\tilde{A}_0 \kappa\, x)\, e^{i\frac{\Delta k}{2}x},$$
$$b = -\frac{\Delta k}{2\kappa} - i\tilde{A}_0 \tanh\,(\tilde{A}_0 \kappa x) \text{ where } A_0^2 = \tilde{A}_0^2 + \frac{\Delta k}{2\kappa}\ .$$

Other solutions to the time-independent problem are given in Refs. [21,22].

Now we consider the time-dependent problem neglecting the group velocity dispersion terms in Eqs. (40) and (41). Since second-order time derivatives are neglected, we may perform the transformation of variables

$$a = \tilde{a} \exp\left[i\ \frac{\Delta k}{\ell}\left(t - \frac{x}{v_1}\right)\right]\ ,$$
$$b = -i\ \tilde{b} \exp\ \left[i\ \frac{\Delta k}{\ell}\left(t - \frac{x}{v_2}\right)\right],$$

with

$$\ell = 1/v_2 - 1/v_1,\quad v_2 = v_g(2\omega), v_1 = v_g(\omega)\ .$$

If we also define characteristics

$$\eta = t - x/v_2,\quad s = t - x/v_1,$$

we obtain

$$-\ell\ \frac{\partial \tilde{a}}{\partial \eta} = \left(\frac{\partial}{\partial x} + \frac{1}{v_1}\ \frac{\partial}{\partial t}\right)\tilde{a} = -\kappa \tilde{a}^* \tilde{b}\ , \tag{48}$$

$$\ell \frac{\partial \tilde{b}}{\partial s} = \left(\frac{\partial}{\partial x} + \frac{1}{v_2}\ \frac{\partial}{\partial t}\right)\tilde{b} = \kappa \tilde{a}^2\ . \tag{49}$$

For many specific boundary value problems, the system (48) and (49) may be solved by the inverse scattering method [23] or from Bäcklund transformations [24].

Both methods show that due to the type of nonlinearity, under certain conditions, the pulses may exchange energy and change shape so that they are not true solitons. Bäcklund transforms also generalize solutions where there is no exchange energy. Under these conditions, the pulses travel with the same soliton velocity, but one pulse may be a soliton and the other a shock wave. We will discuss this type of solution using simpler methods. First, we assume that both waves propagate with the same velocity $v_s$, so that the reduced time $t_r = t - x/v_s$. From this, the system (48) and (49) reduces to the stationarylike coupled ordinary differential equations with $\tilde{a} = \tilde{a}^*$:

$$\left(\frac{1}{v_1} - \frac{1}{v_s}\right)\frac{\partial}{\partial t_r}\tilde{a} = -\kappa \tilde{a}\tilde{b}\ , \tag{50}$$

$$\left(\frac{1}{v_2} - \frac{1}{v_s}\right)\ \frac{\partial}{\partial t_r}\ \tilde{b} = \kappa\ \tilde{a}^2\ . \tag{51}$$

The following conservation law is obtained:

$$\left(\frac{v_s - v_1}{v_1\ v_s}\right)\tilde{a}^2 + \left(\frac{v_s - v_2}{v_2\ v_s}\right)\tilde{b}^2 = \left(\frac{v_s - v_1}{v_s\ v_1}\right)\tilde{a}_0^2$$

with $\tilde{a}_0 = \text{const}$, and is used to obtain the solutions

$$\tilde{a} = \tilde{a}_0 \text{sech}\,(\tilde{a}_0\ \kappa_1\ t_r)\ ,$$
$$\tilde{b} = \frac{v_2\ v_s\ \kappa}{(v_s - v_2)\kappa_1}\ \tilde{a}_0 \tanh\,(\tilde{a}_0\ \kappa_1\ t_r),$$

where

$$\kappa_1^2 = (\kappa\ v_s)^2\left[\frac{v_1\ v_2}{(v_s - v_1)(v_s - v_2)}\right]\ .$$

These solutions were obtained by Armstrong *et al.* [25]. Bäcklund transform also generalize these results [24]. The field $\tilde{a}$ propagates as a true soliton, but the field $\tilde{b}$ propagates as a shocklike solitary wave. These results are slight generalizations of the time-independent case. Note that the Duffing equation for $\tilde{a}$ may be obtained by squaring and differentiating Eq. (50). Here the independent variable is the reduced time $t_r$.

Another interesting problem arises if we go back to the system (48) and (49), using the characteristics $\eta, s$. If we assume $\tilde{a} = \tilde{a}^* = e^U$ and differentiate Eq. (48) by $\frac{\partial}{\partial s}$, we obtain the Liouville equation [26]

$$\frac{\partial^2\ U}{\partial\ \eta\ \partial\ s} = \frac{\kappa^2}{\ell^2}\ e^{2U}\ . \tag{52}$$

The solution is given in [27] using Bäcklund transforms to convert Eq. (52) to the linear wave equation. The solution of (52)

$$\tilde{a}^2 = e^{2U} = \frac{\ell^2}{2\ \kappa^2}\ \frac{f'(\eta)\ g'(s)}{[\ f(\eta)/2 + g(s)]^2}$$

is composed of the solutions of the linear equation consisting of $f(\eta)$ and $g(s)$. We find that the quadratic nonlinear system (48) and (49) is robust enough to induce cubic, inverse cubic, and exponential nonlinearities, depending on the signaling problem.

For sufficiently short pulses, we must include group velocity dispersion effects. We examine two particular signaling problems that include these effects. In order to simplify our analysis, we transform Eqs. (40) and (41) in the following manner: we let $a = \underline{a}, b = \underline{b}\ e^{-i\Delta k x}$, $x = \tilde{x}$, and $s = t - k'(\omega)x$. This results in

$$i\frac{\partial \underline{a}}{\partial x} - \frac{k''(\omega)}{2}\ \frac{\partial^2}{\partial s^2}\ \underline{a} = \kappa\ \underline{a}^* \underline{b}\ , \tag{53}$$

$$i\left(\frac{\partial}{\partial x} + \ell \frac{\partial}{\partial s}\right)\ \underline{b} - \frac{k''(2\omega)}{2}\ \frac{\partial^2}{\partial s^2}\ \underline{b} + \Delta k \underline{b} = \kappa\ \underline{a}^2\ . \tag{54}$$

Note that $\ell = 1/v_2 - 1/v_1$ is the pulse walk-off parameter. As an example, we seek special solutions of the form

$$\underline{a} = \tilde{a}(\tilde{s})\ e^{i(hx + rs)}\ ,\ \underline{b} = \tilde{b}(\tilde{s})\ e^{2i(hx+rs)},$$

with $\tilde{s} = s - x/v_s$ and $v_s$ denoting the solitary-wave velocity. Let $k''(\omega) < 0$ and $k''(2\omega) > 0$. The variables $\tilde{a}, \tilde{b}$ and the parameters $h, r, v_s$ are real. If we choose $r = \frac{1}{v_s\ k''(\omega)}$ and $\frac{1}{v_s} = \frac{\ell\ k''(\omega)}{k''(\omega) + 2\ k''(2\omega)}$ and substitute the special solutions into Eqs. (53) and (54), we obtain real coupled ordinary differential equations with respect to the independent variable $\tilde{s}$:

$$\frac{1}{2}\, k''(\omega)\, \tilde{a}'' - q_1\, \tilde{a} = \kappa\, \tilde{a}\, \tilde{b}, \tag{55}$$

$$-\frac{1}{2} k''(2\omega)\, \tilde{b}'' + q_2\, \tilde{b} = \kappa\, \tilde{a}^2, \tag{56}$$

where $q_1 = [\frac{r^2}{2}\, k''(\omega) + h]$ and $q_2 = [2r^2\, k''(2\omega) - 2r\ell - 2h + \Delta k]$. We are free to choose parameter $h$ to obtain various solutions. For example, we may show that

$$\tilde{a} = a_0 \operatorname{sech} \delta\tilde{s} \tanh \delta\tilde{s} = -\frac{a_0}{\delta}\, \frac{d}{d\tilde{s}}\, (\operatorname{sech} \delta\tilde{s})$$

and

$$\tilde{b} = -a_0 \operatorname{sech}^2 \delta\tilde{s} = -\frac{a_0}{\delta}\, \frac{d}{d\tilde{s}}\, (\tanh \delta\tilde{s})$$

provided

$$\delta^2 = \frac{\kappa\, a_0}{3\, k''(\omega)\sqrt{k''(2\omega)}} = \frac{2q_1}{k''(\omega)} = -\frac{q_2}{k''(2\omega)}$$

and $\delta^2$ is positive. If $r = \Delta k = \ell = 0$ and $|k''(\omega)| = |k''(2\omega)|$, then $q_1 = h$ and $q_2 = -2h$ so that $\delta^2 = \frac{\kappa\, a_0}{3\, k''(\omega)^{3/2}} = \frac{2h}{k''(\omega)}$. The parameter $h$ depends on the amplitude $a_0$ which is arbitrary. However, for the preceding case, with nonzero $r, \ell, \Delta k$, we find that $h$ and $a_0$ are determined. Another example arises if we let $k''(\omega) > 0$; then $q_1 = h - r^2\, k''(\omega)/2$. We may choose $h$ so that $-\frac{q_1}{k''(\omega)} = \frac{q_2}{k''(2\omega)}$. Equations (55) and (56) may then be reduced to a single equation of the form

$$\overset{\approx}{a}{}'' - \frac{q_2\, \overset{\approx}{a}}{k''(2\omega)} = \frac{-\kappa\, \overset{\approx}{a}{}^2}{k''(\omega)k''(2\omega)^{1/2}}.$$

One solution to this equation is

$$\overset{\approx}{a} = \frac{3}{2}\, \frac{q_2\, k''(\omega)}{\kappa\, \sqrt{k''(2\omega)}}\, \operatorname{sech}^2 \left(\frac{q_2}{4k''(2\omega)}\right)^{1/2} \tilde{s}\ .$$

Another interesting solution to Eqs. (53) and (54) arises in the nondepleted pump approximation, within the efficient phase-matching regime. The second-harmonic field $\underline{b}$ is assumed to be weak compared to the fundamental field $\underline{a}$. Hence we may apply perturbation theory to Eqs. (53) and (54) or the system (40) and (41). Remember that MMS theory was used to derive Eqs. (40) and (41) from the medium Eqs. (2) and (5). We apply MMS again to solve Eqs. (53) and (54). Here we assume

$$\underline{a} \sim 1, \quad \underline{b} \sim \epsilon, \quad \kappa \sim \epsilon\ .$$

We assume that $\underline{a}$ is modulated in the form $\underline{a} = \tilde{a} \exp i[R(\Omega)x - \Omega s]$, where $\tilde{a}$ depends on the slower time, longer distance scales. $O(1)$ perturbation theory shows that the dispersion relation $R(\Omega) = k''(\omega)\Omega^2/2$ is obeyed between the new wave number $R(\Omega)$ and the new frequency $\Omega$. Thus $\underline{a}$ depends on two frequencies $\omega$ and $\Omega$.

The $O(\epsilon)$ perturbation shows that

$$\tilde{b} = \frac{\kappa\, \tilde{a}^2\, e^{2i[R(\Omega)x - \Omega s]}}{[\Delta k + 2\ell\Omega + 2k''(2\omega)\Omega^2 - 2R(\Omega)]}$$

and that $\tilde{a}$ travels with the inverse group velocity $R'(\Omega) = k''(\omega)\Omega$ so that $\tilde{a} = \tilde{a}(s_r)$ with reduced time $s_r = s - R'(\Omega)x$. Second-order perturbation theory resulting from MMS shows that the envelope $\tilde{a}$ is self-modulated because it satisfies the cubic nonlinear Schrödinger equation

$$i\, \frac{\partial}{\partial x}\, \tilde{a} - \frac{1}{2}\, k''(\omega)\, \frac{\partial^2}{\partial\, s_r^2}\, \tilde{a} - \tilde{\beta}\, \tilde{a}^3 = 0\ , \tag{57}$$

with the nonlinear coefficient defined as

$$\tilde{\beta} = \frac{8\omega^4[d^{2\omega)}]^2}{\{\Omega^2[2k''(2\omega) - k''(\omega)] + 2\ell\Omega + \Delta k\}\; c^4 k^2(\omega) k(2\omega)}\ . \tag{58}$$

A solution to Eq. (57) is

$$\tilde{a} = a_0\, \operatorname{sech}\left(\left|\frac{\tilde{\beta}}{k''(\omega)}\right|^{1/2} a_0 s_r\right)\, e^{-i\tilde{\beta} a_0^2 x/2} \tag{59}$$

for initial amplitude $a_0$. The nonlinear Schrödinger equation (57), the effective intensity-dependent refractive index coefficient $\tilde{\beta}$ [Eq. (58)], and the solution (59) may be compared to the Schrödinger equation derived directly from the medium Eqs. (2) and (5) as was done in [4,10,11]. For our purposes, we may utilize Eqs. (21), (22), and (27), but now we must set the dc field $A = 0$. We notice the $\tilde{\beta}$ and $\beta$ coefficients are different and that $\tilde{\beta}$ is parametrized by two frequencies $\omega$ and $\Omega$ whereas $\beta$ is parametrized by only $\omega$. The characteristics $(s_r, \tilde{s})$ are also different and for $A = 0$, $\tilde{s} = s$ in Eq. (28).

The Schrödinger equations (57) and (21) were derived under different asymptotic regimes. Equation (57) was derived from Eqs. (40) and (41) for two fields with efficient $O(\epsilon)$ phase matching. Equation (21) with zero dc field $A$ was derived from a single input field propagating in a medium described by Eqs. (2) and (5) with $O(1)$ phase matching. The total $O(1)$ optical field for system (40) and (41) is described by Eq. (39) with

$$E = \frac{\tilde{a}}{\sqrt{k(\omega)}}\, e^{i[R(\Omega)x - \Omega s]}\, e^{i\theta} + \frac{\kappa \tilde{a}^2\, e^{-i\Delta k x}}{\sqrt{k(2\omega)/2}}\, \frac{e^{i2[R(\Omega)x - \Omega s]}\, e^{i[k(2\omega)x - 2\omega t]}}{[\Delta k + 2\ell\Omega + 2k''(2\omega)\Omega^2 - 2R(\Omega)]} + \text{c.c.}, \tag{60}$$

with $\tilde{a}$ given by Eq. (59). Note that because of $\Omega$, the fundamental harmonic field in Eq. (60) has an extra phase factor dependent on $\Omega$, which would be absent for the field derived from Eq. (21). Equations (40) and (41) are no longer valid if $\Delta k \sim 1$. Other equations describe the $a, b$ fields.

## VII. $O(1)$ PHASE-MISMATCH EQUATIONS

Again MMS will be useful in deriving unexpected results. In this regime, the quadratic nonlinearities are not resonant to the linear part of Eq. (5) and are not secular. They remain bounded source terms for the $O(\epsilon)$ fields. This is because the phase mismatch $\Delta k \sim 1$. This may be illustrated by squaring $\underline{a}\ e^{i[k(\omega)x-\omega t]}$.

We then have $\underline{a}^2\ e^{i[2k(\omega)x-2\omega t]}$. We may add and subtract $k(2\omega)x$ to the phase factor. Thus we have $\underline{a}^2\ e^{-i\Delta kx}\ e^{i[k(2\omega)x-2\omega t]}$. If we further assume $\Delta k = \epsilon p$, then $\Delta kx = \epsilon px = px_1$. Thus the $\Delta kx$ phase factor depends on the longer distance scale $x_1 = \epsilon x$. The remaining phase factor depends on the original scales and is secular to the linear left-hand side of Eq. (5) and thus included in Eq. (41). When $\Delta k \sim 1$ the phase term $2k(\omega)x - 2\omega t$ cannot be changed into $k(2\omega)x - 2\omega t$ on the fastest time scale in MMS theory. Therefore, quadratic nonlinearities remain bounded for large phase mismatch. At second-order perturbation, the $O(\epsilon)$ nonresonant quadratic nonlinearities induce secular cubic nonlinearities by "wave mixing" as was shown in [10,11]. For inefficient phase matching, the field pulse widths and intensities are chosen to balance these induced nonlinearities with dispersion so that the application of MMS theory to the two fields of Eq. (39) propagating in a quadratic medium described by Eqs. (2) and (5) or (3) and (6), leads to [4]

$$i\ \frac{\partial}{\partial x}\ a - \frac{1}{2}\ k''(\omega)\ \frac{\partial^2}{\partial s^2}a$$

$$-\bar{\beta}(\omega)\ |a|^2 a - \nu_1\ |b|^2 a = -\frac{i\sigma(\omega)}{2}a\ , \quad (61)$$

$$i\left(\frac{\partial}{\partial x} + \ell\ \frac{\partial}{\partial s}\right) b - \frac{k''(2\omega)}{2}\ \frac{\partial^2}{\partial s^2}b$$

$$-2\bar{\beta}(2\omega)|b|^2 b - \nu_2\ |a|^2 b = -\frac{i\sigma(2\omega)}{2}b\ . \quad (62)$$

Equations (61) and (62)) represent an asymptotic wave-wave process beyond cascading. Here we are in the coordinate frame with reduced time $s = t - x/v_1$ and $\ell$ is the walk-off parameter defined for Eqs. (53) and (54). The following coefficients are defined for $n = 1, 2$:

$$\bar{\beta}(n\omega) = \frac{\beta(n\omega)}{k(n\omega)}\ [1 + g(n\omega)]\ , \quad (63)$$

with $\beta(\omega)$, $g(\omega)$, and $\sigma(\omega)$ given, respectively, in Eqs. (22), (23), and (25). We may substitute $2\omega$ for $\omega$ in Eqs. (23) and (25); however, we prefer to redefine $\beta(2\omega)$ as

$$\beta(2\omega) = \frac{4}{3}\ \frac{[d^{(2\omega)}]^2(\omega_0^2 - 24\omega^2)(\omega_0^2 - \omega^2)^4}{\omega_p^2\omega_0^2(\omega_0^2 - 4\omega^2)^2 c^2 k(2\omega)} \quad (64)$$

because we keep $d^{(2\omega)}$ in Eq. (26) dependent only on $\omega$. That is, we keep the same value of $d^{(2\omega)}$ in Eqs. (22) and (64). The coefficients $\nu_1, \nu_2$ are given as

$$\nu_1 = \nu + G\ , \quad (65)$$

$$\nu_2 = 2\left[\nu + \frac{v_1^2}{v_2^2}\ \left(\frac{V^2 - v_2^2}{V^2 - v_1^2}\right) G\right]\ , \quad (66)$$

with $V$ the velocity of the $O(\epsilon)$ rectified field that comes in at $O(\epsilon^3)$ perturbation theory. The $g(n\omega)$ and $G$ terms include the higher-order corrections as shown in [11]. The coefficients $(G, \nu)$ depend on $\omega$ in the following manner:

$$G = \frac{16\ \omega^2(\omega_0^2 - \omega^2)^2\ [d^{(2\omega)}]^2}{c^4\ k(\omega)\ k(2\omega)\ \omega_0^4\ \left[\frac{1}{v_2^2} - \frac{1}{V^2}\right]}\ , \quad (67)$$

$$\nu = \frac{4[d^{(2\omega)}]^2(\omega_0^2 - \omega^2)^2}{c^2\ k(\omega)\ k(2\omega)} \left\{ \frac{-4\omega^2}{\omega_p^2\ (\omega_0^2 - \omega^2)} - \frac{4\omega^2}{\omega_p^2(\omega_0^2 - 9\omega^2)} - \frac{4\omega^2}{\omega_p^2\omega_0^2} + \frac{4\omega^4}{(\omega_0^2 - \omega^2)D[k(2\omega) - k(\omega); \omega]} + \frac{32\ \omega^4}{(\omega_0^2 - 9\omega^2)D[k(2\omega) + k(\omega); 3\omega]} \right\}\ . \quad (68)$$

Note that the dispersion relation $D[k(n\omega); n\omega]$ was defined in Eq. (14) and is zero if the wave number $k$ is a function of the frequency that is used. In Eq. (68), we have sums and differences of wave numbers substituted for $k(n\omega)$ and hence the function $D$ is not zero as in Eq. (14).

The system of equations (61) and (62) derived from the two fields of Eq. (39) in the quadratic nonlinear medium with $O(1)$ phase matching is analogous to the cross-phase-modulation equations of fiber optics [17]. Solutions originally developed for fiber optics can be adapted to this asymptotic regime. Note that if either $a$ or $b$ is set to zero, we recover the single nonlinear Schrödinger equation derived in [4,10,11]. Here the resulting equation is equivalent to Eq. (21) for zero dc field $A$. If we consider the nondepleted pump approximation, where $b$ is small $[O(\epsilon)]$ compared to $a$, we actually solve the boundary value problem that was given in [10,11]. The only difference is that Eq. (62) for $b$ would appear at a higher perturbation level where the field $a$ would appear as a known function. In [10,11], it was shown that $b$ propagated unchanged along the characteristic with reduced time $(t - x/v_2)$, where we neglected the group velocity dispersion and nonlinear terms in Eq. (62). Therefore, Eq. (62) provides on $O(\epsilon)$ correction where $b$ will no longer be constant along the characteristic. Of course, we neglected damping terms.

If we also neglect the second-order reduced time derivatives associated with group velocity dispersion, we obtain the conservation laws

$$\frac{\partial}{\partial x}\ |a|^2 = \left(\frac{\partial}{\partial x} + \ell\frac{\partial}{\partial s}\right)\ |b|^2 = 0\ .$$

These imply the solutions

$$a = \bar{a}(s)\ e^{i\phi_1}\ ,\ b = \bar{b}\ (s - \ell x)\ e^{i\phi_2}\ ,$$

with

$$\phi_1 = \bar{\beta}(\omega)|\bar{a}(s)|^2 x + \nu_1 \int_0^x |\bar{b}(s - \ell x')|^2 \, dx'$$

and

$$\phi_2 = 2\bar{\beta}(2\omega)\ |\bar{b}(\bar{s})|^2 x + \nu_2 \int_0^x |\bar{a}(\bar{s} + \ell x')|^2 \, dx' \ ,$$

with $\bar{s} = s - \ell x$. The amplitudes $\bar{a}(s), \bar{b}(\bar{s})$ are constant along characteristics $s, \bar{s}$. The first terms in the phases represent self-phase-modulation terms whereas the second terms represent cross phase modulation. These solutions are analogous to those in fibers [17] and may exhibit asymmetrical spectral broadening. The pulse shapes remain unchanged as in fibers.

We now would like to examine Eqs. (61) and (62) with the group velocity dispersion terms. However, damping terms are still set to zero. It must be noted that for $v_1 = \pm v_2, k''(\omega) = k''(2\omega)$, and $\bar{\beta}(\omega) = 2\bar{\beta}(2\omega) = \nu_1 = 2\nu_2$ the coupled equations are solved using the inverse scattering transform [28]. A popular numerical solution is the split-step Fourier method. Here we will try to obtain some special analytic solutions. We assume solutions of the form

$$a = \underline{a}(\tilde{s}_1)\ e^{i(h_1 x + r_1 s)}\ ,\ b = \underline{b}(\tilde{s}_1)\ e^{i(h_2 x + r_2 s)},$$

with reduced time $\tilde{s}_1 = s - \lambda x$ and $\lambda$ the inverse soliton velocity. We choose parameters $r_1, r_2, h_1, h_2$ so that

$$r_1 = -\lambda/k''(\omega), \quad r_2 = (\ell - \lambda)/k''(2\omega)\ ,$$
$$h_1 = \frac{k''(\omega)}{2}\ r_1^2 + \frac{k''(\omega)}{2}\ h_1^*\ ,$$
$$h_2 = \frac{k''(2\omega)}{2}\ r_2^2 - \ell\ r_2 + \frac{k''(2\omega)}{2}\ h_2^*.$$

The parameters $h_1, h_2$ are respectively expressed in terms of $h_1^*, h_2^*$. This leads to the following coupled real ordinary differential equations with independent variable $\tilde{s}_1$:

$$\underline{a}'' + h_1^*\ \underline{a} + \frac{2\bar{\beta}(\omega)}{k''(\omega)}\ \underline{a}^3 + \frac{2\nu_1\ \underline{b}^2\underline{a}}{k''(\omega)} = 0\ , \tag{69}$$

$$\underline{b}'' + h_2^*\ \underline{b} + \frac{4\bar{\beta}(2\omega)}{k''(2\omega)}\ \underline{b}^3 + \frac{2\nu_2\ \underline{a}^2\underline{b}}{k''(2\omega)} = 0\ . \tag{70}$$

For example, we may show that $\underline{a} = a_0$ sech$\delta\tilde{s}_1$ and $\underline{b} = b_0$ sech$\delta\tilde{s}_1$ is a solution to Eqs. (69) and (70) provided

$$h_1^* = h_2^* = -\delta^2, \quad \delta^2 = \frac{\bar{\beta}(\omega)}{k''(\omega)}\ a_0^2 + \frac{2\nu_1 b_0^2}{k''(\omega)}\ ,$$

and

$$\delta^2 = \frac{2\ \nu_2\ a_0^2}{k''(2\omega)} + \frac{2\bar{\beta}(2\omega)b_0^2}{k''(2\omega)}\ .$$

The parameters $\delta^2, a_0^2, b_0^2, \lambda$ must be real and nonnegative. The other parameters must be real. Note that we have two extra parameters to the number of relations or equations between them. We may further specify the solution by choosing the value of one parameter and then expressing the remaining parameters in terms of a favorite parameter. For example, we may set $\lambda = 0$, so that $r_1 = 0$ and $r_2 = \ell/k''(2\omega)$. Then we may determine $\delta, b_0$ in terms of $a_0$, which is arbitrary. Another solution is possible if we choose $h_1 = h_2$ and $\lambda \leq 0$. In this case, $\lambda, \delta, b_0$ may be expressed in terms of $a_0$.

It is noted that a solution to Eqs. (69) and (70) of the form $\underline{a} = a_0$ sech $\delta\tilde{s}_1$ and $\underline{b} = b_0$ tanh $\delta\tilde{s}_1$ also exists if the following relations between parameters hold:

$$h_1^* = -\delta^2 - \frac{2\nu_1}{k''(\omega)}\ b_0^2, \quad h_2^* = -\frac{4\ \bar{\beta}(2\omega)}{k''(2\omega)}\ b_0^2\ ,$$

$$\delta^2 = \frac{\bar{\beta}(\omega)}{k''(\omega)}\ a_0^2 - \frac{2\nu_1}{k''(\omega)}\ b_0^2 = \frac{4\ \bar{\beta}(2\omega)}{k''(2\omega)}\ b_0^2 + \frac{2\nu_2\ a_0^2}{k''(2\omega)}\ .$$

Again we may set $\lambda = 0$ or pick $h_1 = h_2$, depending on our needs. Of course, the real and non-negative restrictions are the same as before. The above solutions are interesting because we did not have to restrict ourselves to equations with equal group velocities ($v_1 = v_2$).

We conclude this section by reconsidering the non-depleted pump approximation in a little more detail where $b$ is small compared to $a$. The approximate solution the field $a$ of Eq. (61) is

$$a = a_0 \text{sech}\left( \left| \frac{\bar{\beta}(\omega)}{k''(\omega)} \right|^{1/2} a_0 s \right)\ e^{-i\bar{\beta}(\omega)\alpha_0^2 x/2}.$$

Therefore, $|a|^2$ now enters as a known function in Eq. (62). The field $b$ is then assumed to be of the form

$$b = \underline{b}(s)\ e^{i(h_2 x + r_2 t)} \text{ with } r_2 = \frac{\ell}{k''(\omega)},$$
$$h_2 = \tfrac{1}{2}\ k''(2\omega) r_2^2 + \frac{2}{k''(2\omega)}\ h_2^*\ .$$

Thus, $\underline{b} = b_0$sech $\delta s$ if

$$h_2^* = -\delta^2 = -\frac{\bar{\beta}(\omega)}{k''(\omega)}\ a_0^2,$$

$$b_0 = \pm \frac{k''(2\omega)\ a_0}{4\ \bar{\beta}(2\omega)}\ \left[\frac{\bar{\beta}(\omega)}{k''(\omega)} - \frac{2\ \nu_2}{k''(2\omega)}\right]^{1/2}\ .$$

For a specific signaling problem we find that the correction to the small second-harmonic field $b$ does not propagate at the group velocity $v_2$ but with $v_1$ of the given field $a$ when nonlinear and dispersion terms are included. We notice that $b$ is proportional to $a$; substituting $b$ back into Eq. (61) does not induce a higher-order nonlinearity. It only changes the nonlinear coefficient $\bar{\beta}$ to $\bar{\beta}$+[const $\nu_2$]. We now see the differences in the nondepleted beam approximation when there is $O(\epsilon)$ phase matching as in Eqs. (40) and (41) versus Eqs. (61) and (62) when there is $O(1)$ phase matching. The dominant fields are governed by a single cubic nonlinear Schrödinger equation. This equation is different for the two asymptotic regimes discussed.

## VIII. SUM FREQUENCY GENERATION

So far, we utilized scalar properties of the medium. We now allow anisotropic effects to occur by choosing waves with different polarizations. As with second-harmonic generation, sum frequency generation may be classified into two asymptotic regimes that depend on whether there is efficient $[O(\epsilon)]$ phase matching or large phase mismatch. We now consider the medium to be an orthorhombic biaxial crystal with class $mm2$ symmetry (such as KTP). The efficient phase-matching regime may be obtained by type-II phase matching for a biaxial crystal in the $y$–$z$ plane. We choose two fields to be polarized in the $y$ direction and one field to be polarized in the $z$ direction. The large-phase-mismatch regime is obtained by propagating all fields with the same polarization chosen to be in the $z$ direction. To simplify the derivations, we assume that the nonlinear coefficients are independent of frequency and Kleinman's symmetry condition is valid; given frequencies such that $\omega_3 = \omega_1 + \omega_2$ and polarizations in the $y$–$z$ plane, it can be shown [29] the nonlinear polarizations have the form

$$P_y^{\rm NL}(\omega_3) = 4\ d_{24}\ [E_y(\omega_1)\ E_z(\omega_2) + E_z(\omega_1)\ E_y(\omega_2)],$$
$$P_z^{\rm NL}(\omega_3) = 4\ d_{32}\ E_y(\omega_1)\ E_y(\omega_2) + 4\ d_{33}\ E_z(\omega_1)\ E_z(\omega_2)\ , \qquad (71)$$

where the $d_{ij}$ are constant. Thus, in deriving the equations we, at first, use the integral equation formalism of the medium described by Eqs. (3) and (6). The nonlinear convolution part Eq. (6) reduces to Eq. (71) because of the above assumptions. We now assume the phase matching $\Delta k = k_2(\omega_1) + k_3(\omega_2) - k_2(\omega_3) \sim \epsilon$. The subscripts 2 and 3 of the wave numbers refer to the fact that the wave numbers are polarized, respectively, in the $y$ and $z$ directions.

## IX. $O(\epsilon)$ PHASE-MISMATCH SUM FREQUENCY EQUATIONS

Here we assume three slowly modulated electric fields that depend on the frequencies $\omega_3 = \omega_1 + \omega_2$ and wave numbers $k_2(\omega_1), k_3(\omega_2), k_2(\omega_3)$ so that

$$E = a\ e^{i[k_2(\omega_1)x - \omega_1 t]} + b\ e^{i[k_2(\omega_3)x - w_3 t]} + u\ e^{i[k_3(\omega_2)x - \omega_2 t]} + \text{c.c.} \qquad (72)$$

Equation (72) is substituted into Eqs. (3), (6), and (71) and MMS perturbation theory is applied. Second-order perturbation theory shows that the slowly modulated fields $a, b, u$ are described by the following coupled equations with zero damping:

$$i\left(\frac{\partial}{\partial x} + k_2'(\omega_1)\ \frac{\partial}{\partial t}\right) a - \frac{k_2''(\omega_1)}{2}\ \frac{\partial^2}{\partial t^2}\ a + \frac{2\ \omega_1^2\ d_{24}}{c^2\ k_2(\omega_1)}\ bu^*\ e^{-i\Delta k x} = 0, \quad (73)$$

$$i\left(\frac{\partial}{\partial x} + k_2'(\omega_3)\ \frac{\partial}{\partial t}\right) b - \frac{k_2''(\omega_3)}{2}\ \frac{\partial^2}{\partial t^2}\ b + \frac{2\ \omega_3^2\ d_{24}}{c^2\ k_2(\omega_3)}\ au\ e^{i\Delta k x} = 0, \quad (74)$$

$$i\left(\frac{\partial}{\partial x} + k_3'(\omega_2)\ \frac{\partial}{\partial t}\right) u - \frac{k_3''(\omega_2)}{2}\ \frac{\partial^2}{\partial t^2}\ u + \frac{2\ \omega_2^2\ d_{32}}{c^2\ k_3(\omega_2)}\ a^* b\ e^{-i\Delta k x} = 0. \quad (75)$$

These equations are well known and expected from quadratic nonlinear media and are obtained from various perturbation methods besides MMS [20,30]. We will not attempt to solve the above equations without approximations or simplifications. At first, we set the group velocity dispersion terms $k_2''(\omega_1) = k_2''(\omega_3) = k_3''(\omega_2) = 0$.

It must be mentioned that the resulting first-order system of equations is solved using the inverse scattering transform [31,32], which is different from that used for second-harmonic generation. The Bäcklund transformation method may also be used [24], which generalizes the results of Armstrong *et al.* [25] for three solitary waves traveling at the same velocity. However, we are interested in making two further simplifications. We set $\Delta k = 0$ and assume $k_2'(\omega_3) = k_3'(\omega_2) = v_2^{-1}$ and $k_2'(\omega_1) = v_1^{-1}$. This results in the reduced system of equations

$$\left(\frac{\partial}{\partial x} + \frac{1}{v_1}\ \frac{\partial}{\partial t}\right)\ a = i\ \frac{2\ \omega_1^2\ d_{24}}{c^2\ k_2(\omega_1)}\ bu^*, \qquad (76)$$

$$\left(\frac{\partial}{\partial x} + \frac{1}{v_2}\ \frac{\partial}{\partial t}\right)\ b = i\ \frac{2\ \omega_3^2\ d_{24}}{c^2\ k_2(\omega_3)}\ au\ , \qquad (77)$$

$$\left(\frac{\partial}{\partial x} + \frac{1}{v_2}\ \frac{\partial}{\partial t}\right)\ u = i\ \frac{2\ \omega_2^2\ d_{32}}{c^2\ k_3(\omega_2)}\ a^* b\ . \qquad (78)$$

It has been shown in [25] that under these conditions, the three-wave equations are equivalent to the self-induced transparency equations for a medium with no inhomogeneous broadening. Thus there exists a correspondence between three-wave processes and a two-level quantum optical system. With minor reinterpretation of the fields it can be shown that Eqs. (76)–(78) are equivalent to semiclassical stimulated Raman scattering equations [33]. Thus there exists a correspondence between three-wave processes and stimulated Raman scattering; the stimulated Raman scattering equations can also be reduced to the Maxwell-Bloch equations, as shown in Ref. [33]. If we let $\bar{\eta} = -\frac{v_1 v_2}{v_1 - v_2}\left(t - \frac{x}{v_2}\right)$,

$$\zeta = \frac{v_1}{v_1 - v_2}\ (t - x/v_1)\ , F = ibu^*,$$

$$\Pi = \frac{\omega_3^2 d_{24}}{k_2(\omega_3)} uu^* - \frac{d_{32}\omega_2^2 bb^*}{k_3(\omega_2)},$$

then Eqs. (76)–(78) transform to

$$\frac{\partial a}{\partial \overline{\eta}} = \alpha_1 F, \quad \frac{\partial F}{\partial \zeta} = -\frac{2\, v_2}{c^2}\, a\, \Pi,$$
$$\frac{\partial \Pi}{\partial \zeta} = \frac{v_2 \alpha_2}{c^2}\, [a^* F + a\, F^*]$$

with

$$\frac{2\, \omega_1^2 d_{24}}{c^2\, k_2(\omega_1)} = \alpha_1, \quad \alpha_2 = \frac{4\, \omega_3^2 \omega_2^2 d_{32} d_{24}}{c\, k_3(\omega_2)\, k_2(\omega_3)} .$$

The above self-induced transparency equations may also be solved by the inverse scattering transform [34]. From these equations, it is noted that we have the following conservation law: $\frac{\partial}{\partial \zeta}(\Pi^2 + \alpha_2 |F|^2) = 0$. This suggests that $\Pi = -\cos\theta$ and $\sqrt{\alpha_2} F = \sin\theta$, where $F = F^*$. This implies the sine-Gordon equation

$$\frac{\partial^2 \theta}{\partial \zeta \partial \overline{\eta}} = \frac{2\, v_2}{c^2} \alpha_1 \sin\theta, \tag{79}$$

with the field $a = \frac{c^2}{2 v_2 \sqrt{\alpha_2}} \frac{\partial \theta}{\partial \zeta}$. Through this two-step transformation, we have now shown that under appropriate conditions, the three-wave process supports the sine-Gordon equation. That is, the quadratic-type nonlinearities induce sinusiodal nonlinearities. It is well known that Eq. (79) is solvable by the inverse scattering method [34] and by Bäcklund transform [27]. The single kink solution for Eq. (79) is

$$\theta = 4\, \tan^{-1} \exp\left( a_0\, \zeta + \frac{1}{a_0}\, \frac{2\, v_2\, \alpha_1}{c^2}\, \overline{\eta} \right),$$

which implies

$$a = \frac{c^2}{v_2 \sqrt{\alpha_2}}\, a_0\, \text{sech}\, \left( a_0\, \zeta + \frac{1}{a_0}\, \frac{2\, v_2\, \alpha_1}{c^2}\, \overline{\eta} \right),$$

where $a_0$ is a constant. This hyperbolic secant solution is different from the other solutions presented because it depends on two reduced times.

Another interesting problem includes the effects of group velocity dispersion on the system (73)–(75). For simplicity, we also set $\Delta k = 0$ and $k_2'(\omega_3) = k_3'(\omega_2) = \frac{1}{v_2}$. In addition, we assume the two fields $a, b$ are $O(1)$ and nondepleted, whereas the field $u$ and the constants $d_{24}, d_{32}$ are $O(\epsilon)$ small. MMS second-order perturbation theory is applied to find how the fields $a, b$ are slowly modulated. The fields $a, b$ are given the form

$$a = \tilde{a}\, \exp i[Rx - \Omega_1 s]\ ,\ b = \tilde{b}\, \exp i[\tilde{R}\, x - \Omega_3 s],$$

with $\tilde{a}, \tilde{b}$ dependent on slower scales. These forms are substituted into Eqs. (73)–(75). Since $u$ is small, at $O(1)$ perturbation theory, Eqs. (73) and (74) are decoupled and linear. This results in the respective dispersion relations for $a, b$ : $R = \frac{k_2''(\omega_1)}{2} \Omega_1^2$ and $\tilde{R} = \ell \Omega_3 + \frac{k_2''(\omega_3)}{2} \Omega_3^2$. At $O(\epsilon)$, Eq. (75) is a linear equation for $u$ with a known nonresonant forcing term dependent on $a^* b$. Thus $u = \alpha_0 \tilde{a}^* \tilde{b}\, \exp i[(\tilde{R} - R)x - (\Omega_3 - \Omega_1)s]$ with reduced time $s = t - x/v_1$ and

$$\alpha_0 = \frac{2\, \omega_2^2 d_{32}}{c^2 k_3(\omega_2) \left[ \tilde{R} - R - \ell(\Omega_3 - \Omega_1) - \frac{k_3''(\omega_2)}{2}(\Omega_3 - \Omega_1)^2 \right]} .$$

Therefore, at second-order perturbation theory, we find from Eqs. (73) and (74) that $\tilde{a}, \tilde{b}$ are described by

$$i \left( \frac{\partial}{\partial x} + k_2''(\omega_1)\Omega_1\, \frac{\partial}{\partial s} \right) \tilde{a} - \frac{k_2''(\omega_1)}{2}\, \frac{\partial^2}{\partial s^2} \tilde{a} + \alpha_1 \alpha_0 |\tilde{b}|^2 \tilde{a} = 0\ , \tag{80}$$

$$i \left( \frac{\partial}{\partial x} + [\ell + k_2''(\omega_3)\Omega_3]\, \frac{\partial}{\partial s} \right) \tilde{b} - \frac{k_2''(\omega_3)}{2}\, \frac{\partial^2}{\partial s^2} \tilde{b} + \alpha_3 \alpha_0 |\tilde{a}|^2 \tilde{b} = 0\ . \tag{81}$$

The constants $\alpha_1 = \frac{2\, \omega_1^2 d_{24}}{c^2 k_2(\omega_1)}$ and $\alpha_3 = \frac{2\, \omega_3^2 d_{24}}{c^2 k_2(\omega_3)}$. Note the absence of self-phase modulation terms $|a|^2 a$ in Eq. (80) and $|b|^2 b$ terms in Eq. (81). Possible solitary waves are obtained if we further assume that the fields have the form

$$\tilde{a} = \underline{a}(\overline{\zeta})\, e^{i(h_1 x + r_1 s_1)}\ ,\ \tilde{b} = \underline{b}(\overline{\zeta})\, e^{i(h_2 x + r_2 s_1)},$$

with $s_1 = s - k_2''(\omega_1)\Omega_1 x$ and $\overline{\zeta} = s_1 - \lambda x$. We choose parameters in the following manner:

$$r_1 = -\lambda / k_2''(\omega_1)\ ,\ h_1 = \frac{k_2''(\omega_1)}{2}\, (r_1^2 + h_1^*)\ ,$$
$$r_2 = \frac{\tilde{\ell} - \lambda}{k_2''(\omega_3)}, \quad h_2 = \frac{k_2''(\omega_3)}{2}\, (r_2^2 + h_2^*) - \tilde{\ell} r_2.$$

The new walk-off parameter $\tilde{\ell} = [\ell + k_2''(\omega_3)\Omega_3 - k_2''(\omega_1)\Omega_1]$. Equations (80) and (81) reduce to the following ordinary differential equations with independent variable $\overline{\zeta}$:

$$\underline{a}'' + h_1^*\, \underline{a} - \frac{2\alpha_1\alpha_0}{k_2''(\omega_1)}\, |\underline{b}|^2 \underline{a} = 0\ , \tag{82}$$

$$\underline{b}'' + h_2^* \, \underline{b} - \frac{2\alpha_2\alpha_0}{k_2''(\omega_3)} \, |\underline{a}|^2 \underline{b} = 0 \; . \tag{83}$$

A solution of the form $\underline{a} = a_0 \text{sech}(\delta\bar{\zeta})$ and $\underline{b} = b_0 \text{sech}(\delta\bar{\zeta})$ exists if the coefficients of the nonlinear terms in Eqs. (82) and (83) are both negative. We find $h_1^* = h_2^* = -\delta^2$ and

$$b_0^2 = \frac{\alpha_3}{\alpha_1} \, \frac{k_2''(\omega_1)}{k_2''(\omega_3)} \, a_0^2 \; , \; \delta^2 = \frac{\alpha_3\alpha_0}{k_2''(\omega_3)} \, a_0^2.$$

Note that $\frac{\alpha_3\alpha_0}{k_2''(\omega_3)} < 0$ so that $\delta^2$ is positive. Solutions may further be specified as in Eqs. (69) and (70) by choosing $\lambda = 0$ or $h_1 = h_2$. Equations (80) and (81) for the non-depleted pump approximation were derived from the sum frequency equations (73)–(75) which were in turn derived from the medium equations (3) and (6). These equations are valid for the $O(\epsilon)$ phase-mismatch regime. We now examine sum frequency equations for large phase mismatch.

## X. $O(1)$ PHASE-MISMATCH SUM FREQUENCY EQUATIONS

The three sum frequency equations derived from medium equations (3) and (6) for $O(1)$ phase mismatch are beyond cascading and will be a generalization of Eqs. (61) and (62) in that cubic nonlinearities will be dominant. In this regime, in order to ensure $O(1)$ phase mismatch, all three fields are polarized in the $z$ direction. The wave number subscript 3 will be dropped. The damping terms are zero. The application of MMS theory under the above conditions yields the following coupled partial differential equations for the slowly modulated fields $a, b, u$ of Eq. (72):

$$i\frac{\partial}{\partial x} \, a - \frac{k''(\omega_1)}{2} \, \frac{\partial^2}{\partial s^2} \, a = -\tilde{\alpha}_0 |a|^2 a - \tilde{\beta}_0 |b|^2 a - \mu_0 |u|^2 a \; , \tag{84}$$

$$i\left(\frac{\partial}{\partial x} + \ell_1 \frac{\partial}{\partial s}\right) \, b - \frac{k''(\omega_2)}{2} \, \frac{\partial^2}{\partial s^2} b = -\tilde{\beta}_1 |b|^2 b - \tilde{\alpha}_1 |a|^2 b - \mu_1 |u|^2 b \; , \tag{85}$$

$$i\left(\frac{\partial}{\partial x} + \ell_2 \frac{\partial}{\partial s}\right) \, u - \frac{k''(\omega_3)}{2} \, \frac{\partial^2}{\partial s^2} u = -\mu_2 |u|^2 u - \tilde{\beta}_2 |b|^2 u - \tilde{\alpha}_2 |a|^2 u \; . \tag{86}$$

The above equations are expressed in terms of coordinates $(x, s = t - x/v_g(\omega_1))$ propagating at the group velocity $v_g(\omega_1)$ of field $a$. The walk-off parameters are given as

$$\ell_1 = \frac{1}{v_g(\omega_2)} - \frac{1}{v_g(\omega_1)} \; , \; \ell_2 = \frac{1}{v_g(\omega_3)} - \frac{1}{v_g(\omega_1)} \; .$$

The following coefficients are defined:

$$\tilde{\alpha}_0 = \frac{4 \, d_{33}^2 \, \omega_1^2}{k(\omega_1)c^4} \left( \frac{4\omega_1^2}{4 \, k^2(\omega_1) - k^2(2\omega_1)} + \frac{8}{\frac{1}{v_g^2(\omega_1)} - \frac{1}{V^2}} \right) \; , \tag{87}$$

$$\tilde{\beta}_0 = \frac{4 \, d_{33}^2 \, \omega_1^2}{k(\omega_1)c^4} \left( \frac{2 \, \omega_3^2}{[k(\omega_1) + k(2\omega_2)]^2 - k^2(\omega_3)} + \frac{2(\omega_2 - \omega_1)^2}{[k(\omega_1) - k(\omega_2)]^2 - k^2(\omega_2 - \omega_1)} + \frac{8}{\frac{1}{v_g^2(\omega_2)} - \frac{1}{V^2}} \right), \tag{88}$$

$$\mu_0 = \frac{4 \, d_{33}^2 \omega_1^2}{k(\omega_1)c^4} \left[ \frac{2(\omega_1 + \omega_3)^2}{[k(\omega_1) + k(\omega_3)]^2 - k^2(\omega_1 + \omega_3)} + \frac{2 \, \omega_2^2}{[k(\omega_1) - k(\omega_3)]^2 - k^2(\omega_2)} + \frac{8}{\frac{1}{v_g^2(\omega_3)} - \frac{1}{V^2}} \right] \; , \tag{89}$$

$$\tilde{\alpha}_1 = \frac{4 d_{33}^2 \omega_2^2}{k(\omega_2)c^4} \left[ \frac{2 \, \omega_3^2}{[k(\omega_1) + k(\omega_2)]^2 - k^2(\omega_3)} + \frac{2[\omega_1 - \omega_2]^2}{[k(\omega_2) - k(\omega_1)]^2 - k^2(\omega_1 - \omega_2)} + \frac{8}{\frac{1}{v_g^2(\omega_1)} - \frac{1}{V^2}} \right] \; , \tag{90}$$

$$\tilde{\beta}_1 = \frac{4 d_{33}^2 \, \omega_2^2}{k(\omega_2)c^4} \left[ \frac{4\omega_2^2}{4k^2(\omega_2) - k^2(2\omega_2)} + \frac{8}{\frac{1}{v_g^2(2\omega)} - \frac{1}{V^2}} \right] \; , \tag{91}$$

$$\mu_1 = \frac{4 \, d_{33}^2 \, \omega_2^2}{k(\omega_2)c^4} \left[ \frac{2(\omega_2 + \omega_3)^2}{[k(\omega_2) + k(\omega_3)]^2 - k^2(\omega_2 + \omega_3)} + \frac{2 \, \omega_1^2}{[k(\omega_2) - k(\omega_3)]^2 - k^2(\omega_1)} + \frac{8}{\frac{1}{v_g^2(\omega_3)} - \frac{1}{V^2}} \right] \; , \tag{92}$$

$$\tilde{\alpha}_2 = \frac{4 d_{33}^2 \omega_3^2}{c^4 k(\omega_3)} \left[ \frac{2(\omega_1 + \omega_3)^2}{[k(\omega_1) + k(\omega_3)]^2 - k^2(\omega_1 + \omega_3)} + \frac{2(\omega_2)^2}{[k(\omega_3) - k(\omega_1)]^2 - k^2(\omega_2)} + \frac{8}{\frac{1}{v_g^2(\omega_1)} - \frac{1}{V^2}} \right] \; , \tag{93}$$

$$\tilde{\beta}_2 = \frac{4 d_{33}^2 \omega_3^2}{c^4 \, k(\omega_3)} \left[ \frac{2 \, (\omega_2 + \omega_3)^2}{[k(\omega_2) + k(\omega_3)]^2 - k^2(\omega_2 + \omega_3)} + \frac{2 \, \omega_1^2}{[k(\omega_3) - k(\omega_2)]^2 - k^2(\omega_1)} + \frac{8}{\frac{1}{v_g^2(\omega_2)} - \frac{1}{V^2}} \right] \; , \tag{94}$$

$$\mu_2 = \frac{4 d_{33}^2 \omega_3^2}{c^4 k(\omega_3)} \left[ \frac{4\omega_3^2}{4k^2(\omega_3) - k^2(2\omega_3)} + \frac{8}{\frac{1}{v_g^2(\omega_3)} - \frac{1}{V^2}} \right] . \tag{95}$$

Solutions are easily obtained if all the pulses are wide enough to neglect the second-order reduced time derivatives (in $s$). For each field, there is one self-phase and two cross-phase-modulation terms. The solutions have the form

$$a = \underline{a}(s)\ e^{i\phi_1}\ ,\ b = \underline{b}(\overline{s})\ e^{i\phi_2}\ ,\ u = \underline{u}(\underline{s})\ e^{i\phi_3},$$

with $\overline{s} = s - \ell_1 x$ and $\underline{s} = s - \ell_2 x$. The phases are

$$\phi_1 = -\tilde{\alpha}_0 |\underline{a}(s)|^2 x - \tilde{\beta}_0 \int_0^x |\underline{b}(s - \ell_1 y)|^2 dy$$
$$-\mu_0 \int_0^x |\underline{u}(s - \ell_2 y)|^2 dy\ ,$$
$$\phi_2 = -\tilde{\beta}_1 |\underline{b}(\overline{s})|^2 x - \tilde{\alpha}_1 \int_0^x |\underline{a}(\overline{s} + \ell_1 y)|^2 dy$$
$$-\mu_1 \int_0^x |\underline{u}[\overline{s} - (\ell_2 - \ell_1) y]|^2 dy,$$

and

$$\phi_3 = -\mu_3 |\underline{u}(\underline{s})|^2 x - \tilde{\alpha}_2 \int_0^x |\underline{a}(\underline{s} + \ell_2 y)|^2 dy$$
$$-\tilde{\beta}_2 \int_0^x |\underline{b}[\underline{s} + (\ell_2 - \ell_1) y]|^2 dy\ .$$

Here we expect spectral broadening phenomena.

The solution changes somewhat if group velocity dispersion terms are included. A solution to Eqs. (84)–(86) is obtained if we assume the following for the fields $a, b, u$:

$$a = \tilde{a}(\zeta_i)\ e^{i(r_1 s + h_1 x)}\ ,$$
$$b = \tilde{b}(\zeta_i)\ e^{i(r_2 s + h_2 x)}\ ,$$
$$u = \tilde{u}(\zeta_i)\ e^{i(r_3 s + h_3 x)},$$

with $\zeta_i = s - \lambda x$. The parameters $r_1, r_2, r_3$ are chosen to be

$$r_1 = -\lambda / k''(\omega_1)\ ,$$
$$r_2 = (\ell_1 - \lambda) / k''(\omega_2)\ ,$$
$$r_3 = (\ell_2 - \lambda) / k''(\omega_3)\ .$$

The parameters $h_1, h_2, h_3$ are rescaled in terms of $h_1^*, h_2^*, h_3^*$ so that

$$h_1 = \frac{k''(\omega_1)}{2}\ r_1^2 + \frac{k''(\omega_1)}{2}\ h_1^*\ ,$$
$$h_2 = \frac{k''(\omega_2)}{2}\ r_2^2 - \ell_1\ r_2 + \frac{k''(\omega_2)}{2}\ h_2^*\ ,$$
$$h_3 = \frac{k''(\omega_3)}{2}\ r_3^2 - \ell_2\ r_3 + \frac{k''(\omega_3)}{2}\ h_3^*\ .$$

The system (84)–(86) is reduced to the following coupled ordinary differential equations (in variable $\zeta_i$):

$$\tilde{a}'' + h_1^*\ \tilde{a} = \frac{2}{k''(\omega_1)}\ \left(\tilde{\alpha}_0\ |\tilde{a}|^2 + \tilde{\beta}_0\ |\tilde{b}|^2 + \mu_0\ |\tilde{u}|^2\right) \tilde{a}, \tag{96}$$

$$\tilde{b}'' + h_2^*\ \tilde{b} = \frac{2}{k''(\omega_2)}\ \left(\tilde{\alpha}_1\ |\tilde{a}|^2 + \tilde{\beta}_1\ |\tilde{b}|^2 + \mu_1\ |\tilde{u}|^2\right) \tilde{b}, \tag{97}$$

$$\tilde{u}'' + h_3^*\ \tilde{u} = \frac{2}{k''(\omega_3)}\ \left(\tilde{\alpha}_2\ |\tilde{a}^2| + \tilde{\beta}_2\ |\tilde{b}|^2 + \mu_2\ |\tilde{u}|^2\right) \tilde{u}. \tag{98}$$

One may show that $\tilde{a} = a_0 \mathrm{sech} \delta\zeta$, $\tilde{b} = b_0 \mathrm{sech} \delta\zeta$, and $\tilde{u} = u_0 \mathrm{sech} \delta\zeta$. The following relations must hold between parameters:

$$h_1^* = h_2^* = h_3^* = -\delta^2\ ,$$
$$\delta^2 = -\frac{1}{k''(\omega_1)}\ \left(\tilde{\alpha}_0\ a_0^2 + \tilde{\beta}_0\ b_0^2 + \mu_0\ u_0^2\right)\ ,$$
$$\delta^2 = -\frac{1}{k''(\omega_2)}\ \left(\tilde{\alpha}_1\ a_0^2 + \tilde{\beta}_1\ b_0^2 + \mu_1\ u_0^2\right)\ ,$$
$$\delta^2 = -\frac{1}{k''(\omega_3)}\ \left(\tilde{\alpha}_2\ a_0^2 + \tilde{\beta}_2\ b_0^2 + \mu_2\ u_0^2\right)\ .$$

Here again, $\lambda, \delta^2, a_0^2, b_0^2, u_0^2$ must be real and positive. Thus the solution exists for various values of the known coefficients $\tilde{\alpha}_i, \tilde{\beta}_i, \mu_i$, etc. The solution may be further specified if $\lambda = 0$, or $h_1 = h_2$, or $h_1 = h_3$, or $h_2 = h_3$, for example. The above relations are then expressed in terms of one parameter, say, $a_0$. This was just one type of solitary-wave solution presented where all waves propagate at the same velocity $\lambda^{-1}$ or $v_g(\omega_1)$ depending on the initial signaling problem. A numerical method must be applied for different propagation velocities of the wave amplitudes where there are no compensating phase factors for the envelopes $a, b, u$.

## XI. CONCLUSIONS

Several problems with multiple input waves propagating in a quadratic nonlinear medium were discussed. The first problem analyzed was the interaction between an optical field $a$ and a constant electric field $A$. The pulse width and intensity were chosen to balance dispersive and cubic nonlinear terms. This resulted in a nontraditional, asymptotic wave-wave process for quadratic nonlinear materials. The cubic nonlinear Schrödinger equation for field $a$ was obtained where the wave number and group velocity depended on the constant field $A$ as shown in Eqs. (21)–(28). The constant field $A$ was then allowed to slowly vary and short to long wave resonance Eqs. (30) and (32) were derived. Here the field pulse widths and intensities were chosen to balance dispersive and quadratic nonlinear terms. Several solitary-wave solutions were then discussed for near and away from short to long wave resonance.

The interaction of fundamental harmonic and second-harmonic optical waves was then examined. There exist two asymptotic regimes for these optical frequency waves propagating in a quadratic nonlinear medium. These regimes depend upon the the phase-matching efficiency. The first regime occurs when there is $O(\epsilon)$ phase mismatch between the waves. In this regime, quadratic nonlinearities dominate as shown in traditional equations (40) and (41). The balance between dispersion and nonlinearity depends on the field pulse widths and intensities. These quadratic nonlinearities may induce cubic, inverse cubic, and exponential nonlinearities depending on the signaling problem. Solutions were given when group velocity dispersion was neglected and included. It was also shown under what conditions it is possible to derive the Liouville [Eq. (52)] and cubic nonlinear Schrödinger equations [Eq. (57)]. It is noted that the Schrödinger equation (57) is different from Eq. (21) in several respects, the most important being that Eq. (57) was derived from Eqs. (40) and (41). The other Schrödinger equation (21) was derived directly from medium equations (2) and (5), which is an asymptotic wave-wave process beyond traditional three-wave resonance.

The second regime for interacting fundamental and second harmonics occurs when there is $O(1)$ phase mismatch. This results in coupled cubic nonlinear Schrödinger equations that are analogous to cross-phase-modulation equations of fiber optics; the results are displayed in Eqs. (61) and (62). In this case the field pulse widths and intensities were picked so that it was possible to offset dispersive effects with cubic nonlinearities. Again, solutions were given for signaling problems that included and neglected group velocity dispersion. The nondepleted pump approximation was discussed and shown to be consistent with [10,11]. The two-input-wave problem resulting in Eqs. (61) and (62) is a natural generalization of the single-input-wave problem of [10,11] and this is another asymptotic wave-wave process beyond traditional three-wave resonance.

The final set of problems dealt with sum frequency generation. Three interacting optical waves were studied in the two asymptotic phase-mismatch regimes considered for harmonic generation. The first asymptotic regime again involved $O(\epsilon)$ phase mismatch. This resulted in traditional equations (73)–(75), where quadratic nonlinearities dominate. When group velocity dispersion was neglected, it was shown how, under appropriate conditions, the three-wave resonance equations transform to the semiclassical stimulated Raman scattering equations and to the self-induced transparency equations and then the sine-Gordon equation (79). Thus the quadratic nonlinearity may also induce sinusoidal nonlinearities. When group velocity dispersion was included, two waves were assumed to be nondepleted. This resulted in cubic-type Schrödinger equations [Eqs. (80) and (81)]. These are different from Eqs. (61) and (62) because they were derived from Eqs. (73)–(75) and not from the medium equations (2) and (5). Also, the self-phase modulation terms are missing from Eqs. (80) and (81).

The sum frequency equations were then derived for the $O(1)$ phase-mismatch regime. This resulted in wave-wave processes beyond cascading by Eqs. (84)–(86) which are cubic Schrödinger equations and are natural generalizations of (61) and (62) and therefore the three-wave generalization of the single-input-wave problem of [10,11]. Analytic solutions were obtained when group velocity dispersion terms were included and neglected. This resulted respectively in solitary "sech"-type and spectral-broadening-type solutions.

To achieve $O(\epsilon)$ phase mismatch, the sum frequency equations were allowed to have different polarizations via type-II phase matching in the $y$-$z$ plane. Therefore, anisotropic effects were included. The $O(1)$ phase-mismatch equations are best achieved if all waves propagate with the same polarization. It must be noted that diffraction effects may easily be included as shown in [4,10,11]. Thus there are many possible generalizations of the problem discussed. For example, equations for wave guides may be developed that differ slightly from the bulk medium equation in dispersive properties. The evolution equations depend upon the efficiency of the phase matching. $O(\epsilon)$ phase matching results in wave-mixing processes where quadratic nonlinearities dominate. $O(1)$ phase matching results in wave-mixing processes where cubic nonlinearities dominate. This distinction between phase-matching regimes also applies to $N$ input waves where $N > 3$.

Many aspects of multiple-wave problems were covered in this paper. It was interesting to learn how quadratic nonlinearities induce higher-order nonlinearities in the various problems discussed. Applying a self-consistent perturbation theory to nondepleted pump approximations resulted in induced cubic nonlinearities. Nonlinear variable transformations resulted in transcendental nonlinearities. MMS perturbation theory was a useful tool for deriving the various wave interaction equations that occurred in the different asymptotic regimes. It was also helpful for obtaining solutions to the nondepleted pump approximations that arose in the $O(\epsilon)$ and $O(1)$ phase-mismatch regimes. MMS was able to derive equations not usually associated with quadratic nonlinearities because it is a self-consistent technique that can be carried to any perturbation order.

SVEA is only able to derive equations with the lowest-order nonlinearity as was discussed in [4,10,11]. It must be mentioned that Eqs. (40) and (41), and (73)–(75) may also be derived by SVEA since they involve the lowest-order (quadratic) nonlinearity. However, the higher-order induced cubic nonlinearities require MMS. As shown in [4,10,11], the induced cubic nonlinearity arises because of wave mixing of an $O(1)$ field with an $O(\epsilon)$ field at second-order perturbation theory. The order $O(\epsilon)$ field itself is stimulated by the presence of a "squared" $O(1)$ field that acts as a known forcing term at the $O(\epsilon)$ perturbation order. The cascading effect discussed by De Salvo *et al.* [22] is also interpreted as an effect arising from wave mixing between $O(1)$ and $O(\epsilon)$ fields that resulted in the induced higher-order nonlinearities for the nondepleted approximation problems that arose from Eqs. (40), (41), and (73)–(75). This wave-mixing effect is not limited to the traditional three-wave resonance equations (40) and (41) and (73)–(75), but also applies to the medium equations (2) and (5) or (3) and (6), from which the $O(1)$ phase-mismatch regime (beyond cascading) is also constructed. The time-independent, three-wave resonance problem described by Eqs. (42) and (43) may be viewed as wave mixing between two $O(1)$ fields. This resulted in a cubic Duffing oscillator for one of the fields that is consistent with the elliptic integral formalism of [21]. Wave-wave interaction or wave mixing provides a feedback mechanism that is responsible for all nonlinear effects observed including self-modulation effects. It applies to all the evolution equations that were derived from the medium equations (2) and (5), or (3) and (6). As defined by [22], cascading appears to be wave-mixing effects restricted to the subset of traditional three-wave resonance evolution equations (40), (41), and (73)–(75). We have shown that there are asymptotic wave-wave processes beyond cascading and beyond traditional three-wave resonance equations in quadratic materials.

Some interesting analytic solutions to the asymptotic evolution equations were obtained by reducing the complex valued systems of partial differential equations to real coupled ordinary differential equations. This was accomplished by appropriate choices of phase parameters. Compatibility conditions were also given to ensure real solitary wave solutions. This technique enabled us to obtain analytic solutions with nonzero walk-off parameters and nonzero group velocity dispersion.

The quadratic nonlinear medium, described by Eqs. (2) and (5), or (3) and (6), is a rich source of different types of evolution equations. A few asymptotic equations were derived for multiple input waves propagating in a strongly dispersive but weakly nonlinear optical material. There are yet other asymptotic regimes that may be studied. The classical quadratic medium may also behave as a cubic medium or a two-level atomic system coupled to a strong optical field depending on the problem. MMS has proven to be a very useful self-consistent technique for obtaining various asymptotic evolution regimes. The theory presented here may be used to model specific $\chi^{(2)}$ materials in order to design new experiments and interesting pulse-shaping applications beyond harmonic generation.

## ACKNOWLEDGMENT

A.G.K. would like to thank his thesis advisor Julian D. Cole for supporting the research effort spent on optical wave propagation.

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[1] N. Bloembergen, *Nonlinear Optics* (Benjamin/ Cummings, New York, 1965).
[2] Y. R. Shen, *The Principles of Nonlinear Optics* (Wiley, New York, 1984).
[3] A. Yariv, *Quantum Electronics* (Wiley, New York, 1989).
[4] A. G. Kalocsai, Ph.D. thesis, Rensselaer Polytechnic Institute, New York, 1991.
[5] J. D. Cole, *Perturbation Methods in Applied Mathematics* (Blaisdell, New York, 1968).
[6] P. A. Sturrock, Proc. R. Soc. London Ser. A **242**, 277 (1957).
[7] G. Sandri, Ann. Phys. (N.Y.) **24**, 332 (1963).
[8] A. D. D. Craik, *Wave Interactions and Fluid Flows* (Cambridge University Press, New York, 1985).
[9] D. G. Swanson, *Plasma Waves* (Academic, New York, 1989).
[10] A. G. Kalocsai and J. W. Haus, Opt. Commun. **97**, 239 (1993).
[11] A. G. Kalocsai and J. W. Haus, Phys. Rev. A **49**, 574 (1994).
[12] A. Hasagawa, *Optical Solitions in Fibers* (Springer-Verlag, New York, 1989).
[13] A. Davey, J. Fluid Mech. **51**, 477 (1972).
[14] H. Washimi and T. Taniuti, Phys. Rev. Lett. **21**, 209 (1966).
[15] V. E. Zakharov, Zh. Eksp. Teor. Fiz. **62**, 1745 (1972) [Sov. Phys. JETP **35**, 908 (1972)].
[16] G. P. Agrawal, *Nonlinear Fiber Optics* (Academic, New York, 1989).
[17] P. N. Butcher and D. Cotter, *The Elements of Nonlinear Optics* (Cambridge University Press, Cambridge, 1990).
[18] K. Nishikawa, H. Hojo, K. Mima, and H. Ikezi, Phys. Rev. Lett. **33**, 148 (1974).
[19] T. Kawahara, N. Sugimoto, and T. Kakutani, J. Phys. Soc. Jpn. **39**, 1379 (1975).
[20] S. A. Akhamov, A. S. Chirkin, K. N. Drabovich, A. I. Kovrigin, R. V. Khokhrov, and A. P. Sukhorukov, IEEE J. Quantum Electron. **4**, 598 (1968).
[21] J. A. Armstrong, N. Bloembergen, J. Ducuing, and P. S. Pershan, Phys. Rev. **127**, 1918 (1962).
[22] J. De Salvo, D. J. Hagan, M. Shiek Bahae, G. Stegeman, and E. W. Van Stryland, Opt. Lett. **17**, 28 (1992).
[23] D. J. Kaup, Stud. Appl. Math. **59**, 25 (1978).
[24] K. M. Case and S. Chiu, Phys. Fluids **20**, 746 (1977).
[25] J. A. Armstrong, S. S. Jha, and N. S. Shiren, IEEE J. Quantum Electron. **6**, 123 (1970).
[26] G. V. Krivoshchenkov, N. G. Nikulin, and V. I. Stroganov, Opt. Spectrosc. **36**, 335 (1974).
[27] C. Rogers and W. F. Shadwick, *Bäcklund Transformation and Their Applications* (Academic, New York, 1982).
[28] S. V. Manakov, Zh. Eksp. Teor. Fiz. **67**, 543 (1974) [Sov. Phys. JETP **40**, 269 (1975)].
[29] R. Boyd, *Nonlinear Optics* (Academic, New York, 1992).
[30] M. F. Becker, Y. C. Kim, S. R. Gautum, and E. J. Powers, IEEE J. Quantum Electron. **18**, 113 (1982).
[31] V. E. Zakharov and S. V. Manakov, Zh. Eksp. Teor. Fiz. **18**, 413 (1973) [Sov. Phys. JETP Lett. **18**, 243 (1973)].
[32] D. J. Kaup, Stud. Appl. Math. **55**, 9 (1976).
[33] C. M. Bowden and J. Englund, Opt. Commun. **67**, 71 (1988).
[34] M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (Siam, Philadelphia, 1981).